Iterative Solvers for Large Linear Systems

Part Ia: Introduction and Basics

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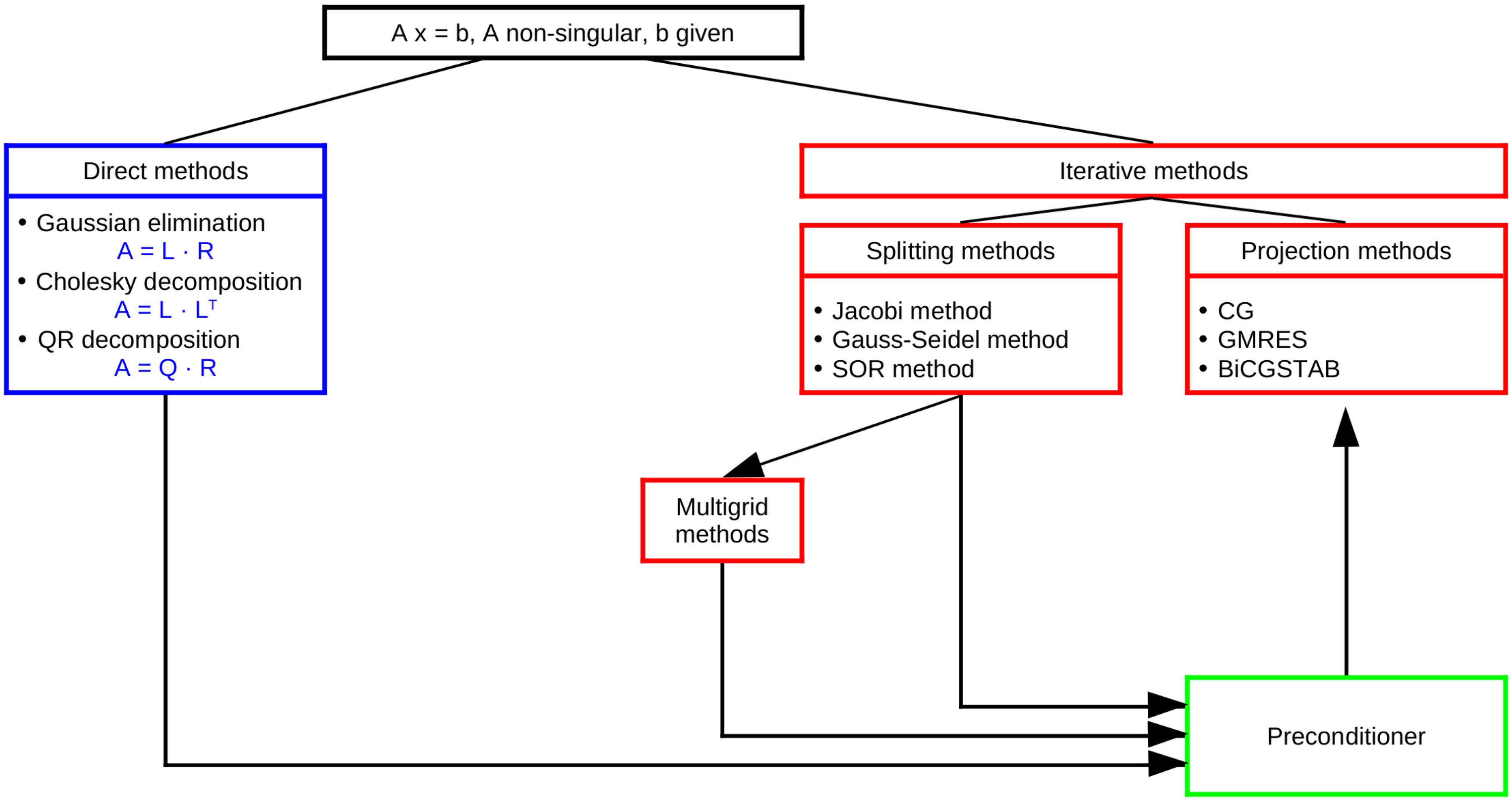
Outline

- Basics of Iterative Methods
- Splitting-schemes
  - Jacobi- u. Gauß-Seidel-scheme
  - Relaxation methods
- Methods for symmetric, positive definite Matrices
  - Method of steepest descent
  - Method of conjugate directions
  - CG-scheme

Outline

- Multigrid Method
  - Smoother, Prolongation, Restriction
  - Twogrid Method and Extension
- Methods for non-singular Matrices
  - GMRES
  - BiCG, CGS and BiCGSTAB
- Preconditioning
  - ILU, IC, GS, SGS, ...

Numerics for linear systems of equations



Fundamentals of Linear Algebra and classical Iterative Solution Methods

- General problem:
  Given: $A \in \mathbb{C}^{n \times n}$ non-singular, $b \in \mathbb{C}^n$
  Sought after: $x \in \mathbb{C}^n$ with $Ax = b$
- Main ideas of Splitting-schemes
  - A trivial approach
- Consistency, convergence and rate of convergence
- Special Splitting-schemes
  - Jacobi-method
  - Gauß-Seidel-method
  - Relaxation schemes
    - SOR-method

Main ideas of Splitting-schemes

Definition: Iterative methods

Choose $x_0 \in \mathbb{C}^n$ arbitrarily and calculate succecively approximations $x_m \in \mathbb{C}^n$ for $x^\star = A^{-1}b$ by means of

$$x_{m+1} = \phi(x_m, b), \quad m = 0, 1, \ldots \ .$$

The method is called linear, if matrices $M, N \in \mathbb{C}^{n\times n}$ exist, such that

$$\phi(x, b) = Mx + Nb.$$

The matrix M is called iteration matrix.

Procedure: Split $A \in \mathbb{C}^{n\times n}$ by means of $B \in \mathbb{C}^{n\times n}$ (non-singular) in the form:

$$A = B + (A - B)$$

Thus, one can write:

$$\begin{aligned} & & Ax &= b \\ &\Longleftrightarrow & Bx + (A - B)x &= b \\ &\Longleftrightarrow & Bx &= (B - A)x + b \\ &\Longleftrightarrow & x &= B^{-1}(B - A)x + B^{-1}b \end{aligned}$$

Main ideas of Splitting-schemes

Choose $x_0 \in \mathbb{C}^n$ arbitrarily and calculated successively

$$x_{m+1} = B^{-1}(B - A)x_m + B^{-1}b, \quad m = 0, 1, \dots .$$

Hence, we get:

$$x_{m+1} = \phi(x_m, b) = Mx_m + Nb$$

with

$$\begin{aligned} M &:= B^{-1}(B - A) \\ N &:= B^{-1} \end{aligned}$$

Conclusion:

Each Splitting scheme is linear

Main ideas of Splitting-schemes

Choose $x_0 \in \mathbb{C}^n$ arbitrarily and calculated successively

$$x_{m+1} = B^{-1}(B - A)x_m + B^{-1}b, \quad m = 0, 1, \ldots .$$

Desired properties of B:

- Good approximation of A (fast convergence)
  - Example: $B = A$

$$\Longrightarrow x_1 = B^{-1}(B - A)x_0 + B^{-1}b = B^{-1}b = A^{-1}b$$

- Easy calculation of the matrix-vector-product $B^{-1}x$ (practicability)
- Less assumptions on A (useability)

A trivial scheme

- Choose $B = I$

$$\begin{aligned} \Longrightarrow \quad M &= I^{-1}(I - A) = I - A \\ N &= I \\ \Longrightarrow \quad x_{m+1} &= (I - A)x_m + b \end{aligned}$$

"+ "	:	no assumptions on A
"+ "	:	$I^{-1}x$ is easy to calculate
"- "	:	bad approximation of A in general

<u>Model problem:</u>

$$\underbrace{\begin{pmatrix} 0.7 & -0.4 \\ -0.2 & 0.5 \end{pmatrix}}_{A:=} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x:=} = \underbrace{\begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix}}_{b:=}$$

- A is non-singular ($\det A = 0.27$) and $x^\star = A^{-1}b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$