Eccentricities in Hanoi Graphs (pr87mo)

Andreas M. Hinz & Ciril Petr

LMU München (Germany) & Univerza v Mariboru (Slovenia)

hinz@math.lmu.de & ciril.petr@gmail.com

SuperMUC-NG Status and Results Workshop, 2023–05–10

La Tour d’Hanoï (Édouard Lucas, 1883)

0. Mathematical Background

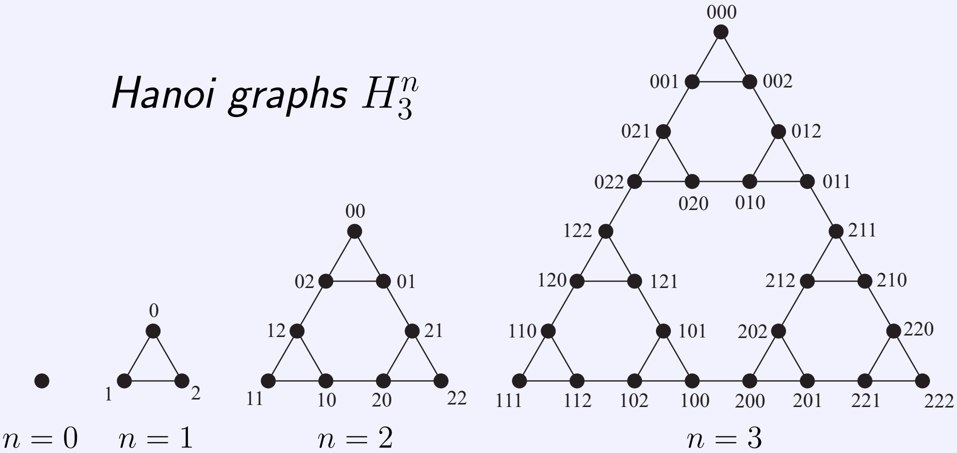
Hanoi graphs with *base* $p \in \mathbb{N}_3$ and *exponent* $n \in \mathbb{N}_0$

$$[p]_0 = \{0, \ldots, p-1\},\ [n] = \{1, \ldots, n\},$$

$$V\left(H_p^n\right) = \{s_n \ldots s_1 \mid s_d \in [p]_0,\ d \in [n]\} \cong [p]_0^n,$$

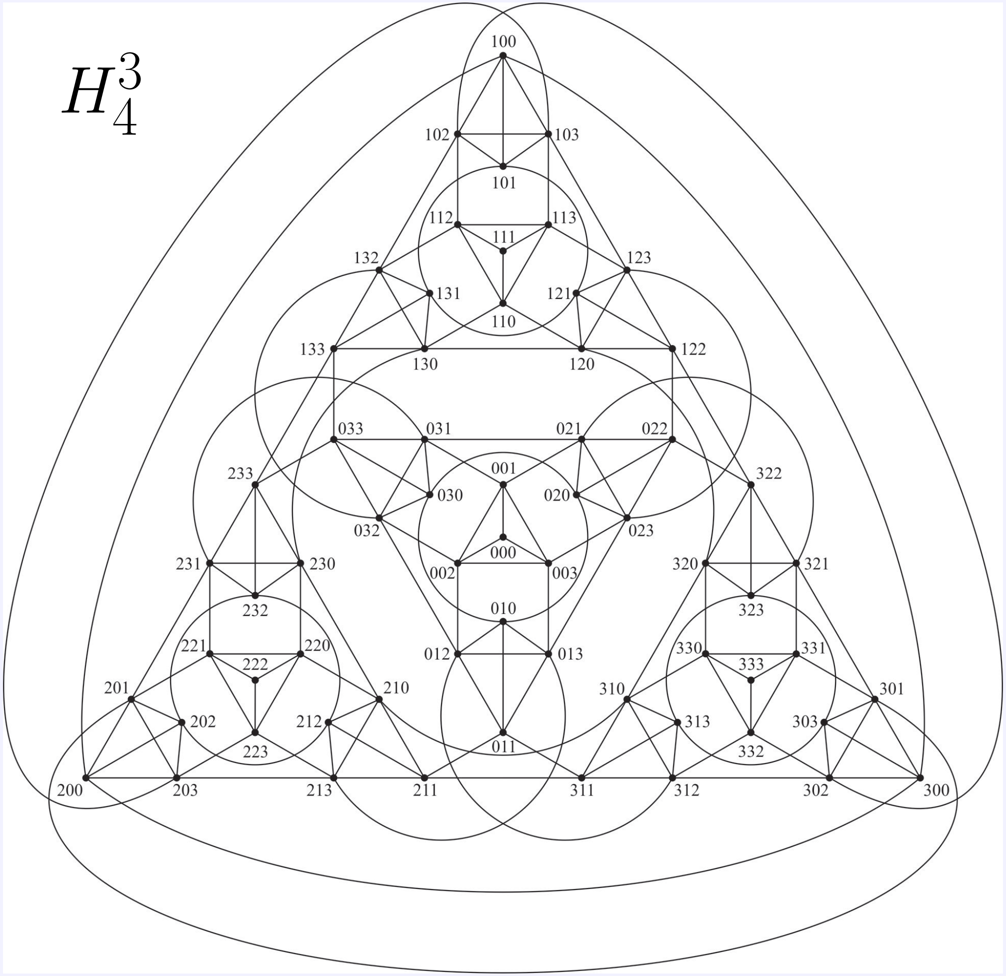
$$E\left(H_p^n\right) = \left\{ \{\underline{s}i\overline{s}, \underline{s}j\overline{s}\} \mid \{i,j\} \in \binom{[p]_0}{2},\ d \in [n],\ \overline{s} \in \left([p]_0 \setminus \{i,j\}\right)^{d-1} \right\}$$

Hanoi graphs H_3^n



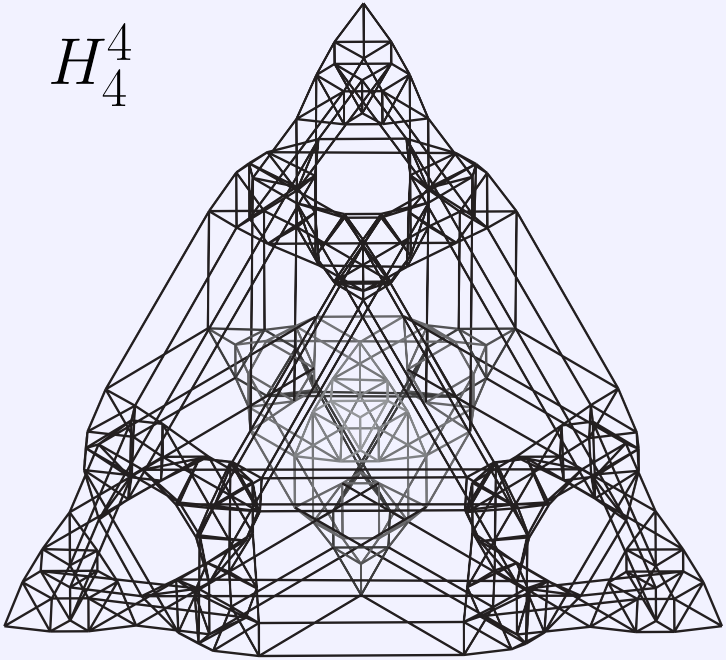
$$\mathrm{d}_3(0^n, 1^n) = \varepsilon_3(0^n) = \mathrm{diam}(H_3^n) = 2^n - 1$$

H_4^3



R. S. Schmid, 2010

H_4^4



C. Petr, 2013

$$|H_p^n| = p^n, \ \|H_p^n\| = \frac{p(p-1)}{4}\,(p^n - (p-2)^n)$$

$$2n-1 \overset{(1)}{\leq} \mathrm{d}_p(0^n, 1^n) \overset{(2)}{\leq} \varepsilon_p(0^n) \overset{(3)}{\leq} \mathrm{diam}(H_p^n) \overset{(4)}{\leq} 2^n-1$$

1. with "$=$" iff $1 \leq n < p$
2. with "$=$" expected, but *Korf's phenomenon* (2004):

   $\mathrm{ex}(n) := \varepsilon_4(0^n) - \mathrm{d}_4(0^n, 1^n) = 1 > 0$ for $n = 15$
3. no case of "$<$" known; in particular,

   $\mathrm{EX}(n) := \mathrm{diam}(H_4^n) - \mathrm{d}_4(0^n, 1^n) = \mathrm{ex}(n)$ so far
4. with "$=$" iff $p = 3$ or $n \leq 2$

Let $\forall\, n \in \mathbb{N}_0 :\ FS_3^n = 2^n - 1$ and for $p \in \mathbb{N}_4$:

$$FS_p^0 = 0,\ \forall\, n \in \mathbb{N} :\ FS_p^n = \min\left\{2FS_p^m + FS_{p-1}^{n-m} \mid m \in [n]_0\right\}\ .$$

Frame-Stewart conjecture: $\mathrm{d}_p(0^n, 1^n) = FS_p^n$

confirmed for $p = 4$: Bousch (2014)

Subtower conjecture: only subtower solutions exist for $0^n \to 1^n$ if $n \geq \binom{p}{2}$.

Korf-Felner conjecture: $\mathrm{ex}(n) > 0$ for $n \geq 20$.

behavior of $\overline{\varepsilon}(H_p^n)/\mathrm{diam}(H_p^n)$

Dudeney-Stockmeyer conjecture: similar optimal strategy for Tower of Hanoi

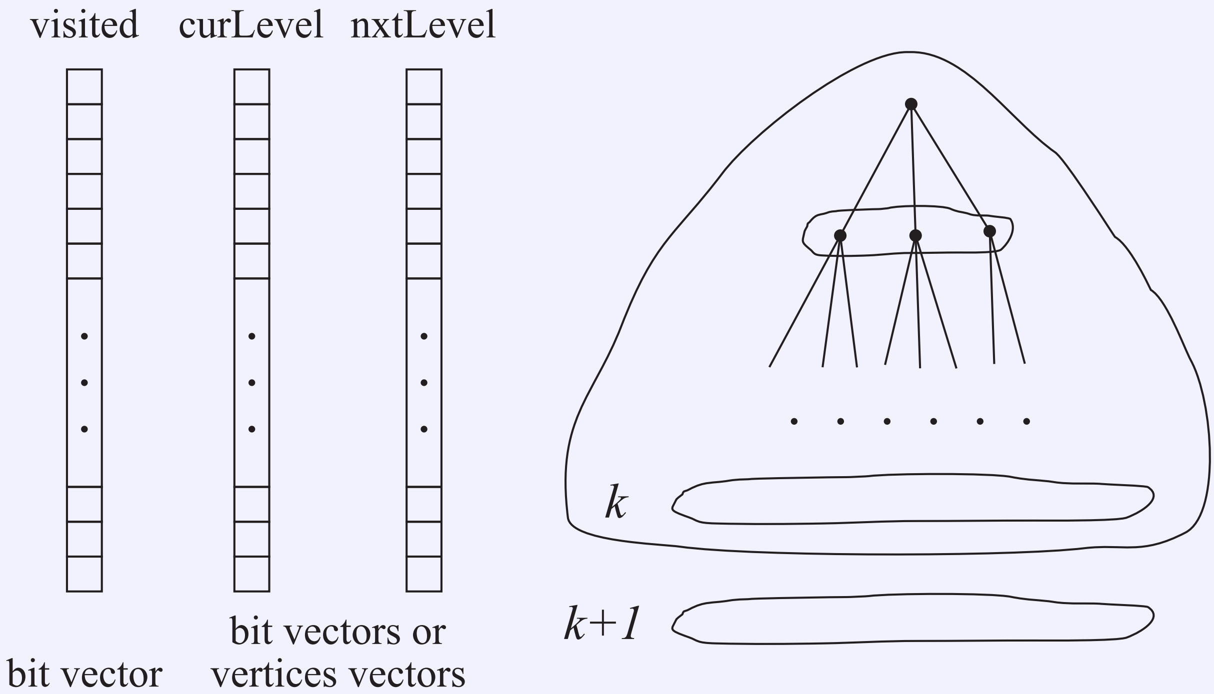
variants like the *Star Tower of Hanoi*; cf. OEIS A291877

Linear Tower of Hanoi for $p \geq 4$; cf. OEIS A160002

What the BFS algorithm offers

- distances
- Korf phenomenon
- Frame-Stewart conjecture
- eccentricities (radius, center, diameter, periphery)
- generating all shortest paths
- analyzing movements of the largest or any other disc

BFS and data structures in internal memory



Many approaches and limitations to implement BFS (1/2)

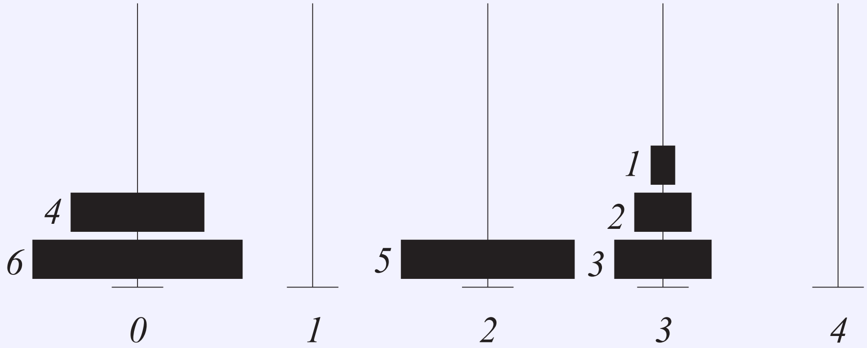
- for small p, n using RAM
- limits on 32 bit architectures
- also on 64 bit architectures arrays are limited, but can be splitted into many pieces
- internal memory enables direct addressing, but is limited
- external memory is usualy file system, by nature sequential

Many approaches and limitations to implement BFS (2/2)

- vertex representation n-tuples, number in p base, 2 bits for each disc in H_4^n
- unique starting vertices, using representatives of equivalence classes
- sorted non-starting pegs
- Delayed Duplicate Detection (DDD BFS)
- Frontier Search DDD BFS
- DDD without sorting

State representation

p=5, n=6



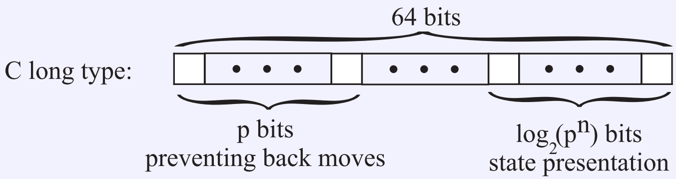
r=(0,2,0,3,3,3)

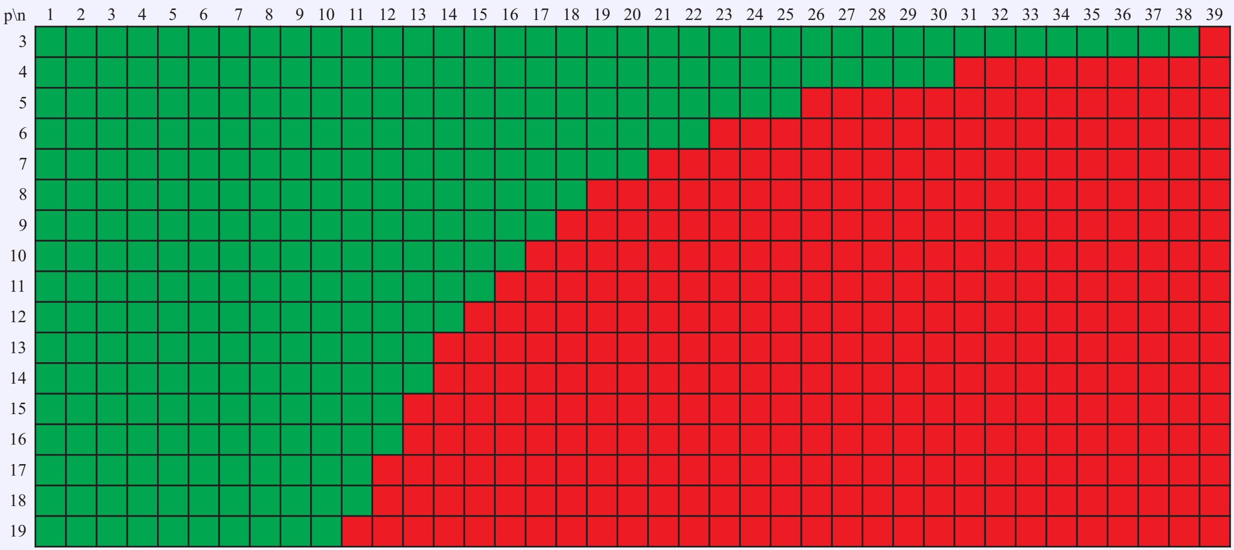
0 * 5^5 + **2** * 5^4 + **0** * 5^3 + **3** * 5^2 + **3** * 5^1 + **3** * 5^0

$1343_{(10)}$

$10100111111_{(2)}$

Current implementation limits

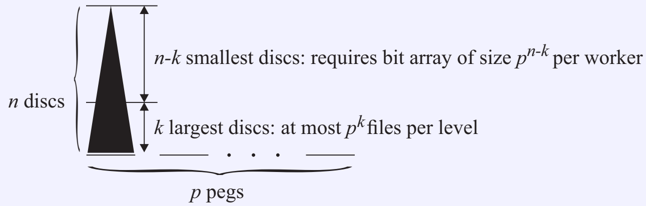


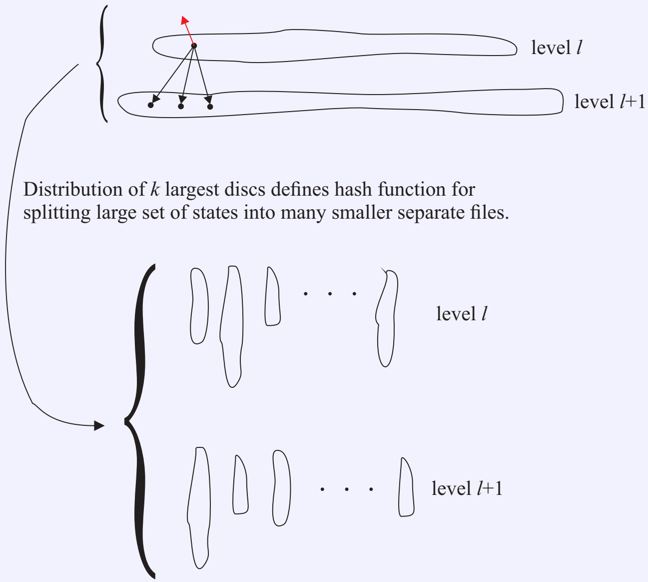


$$p + \log_2(p^n) > 2^{64}$$

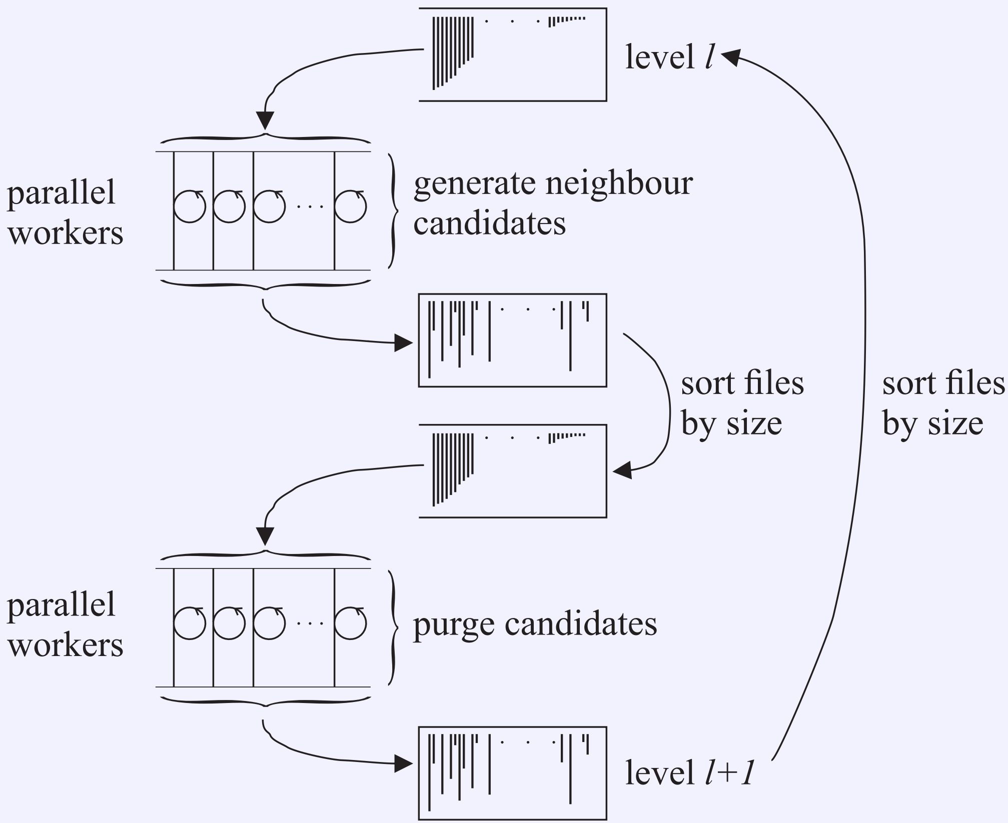
```
procedure DDD_BFS(G, r)
G graph
r root vertex {r ∈ V(G)}
level ← −1; nxtLevel ← {r}
while nxtLevel ≠ {}
    level ← level + 1
    curLevel ← nxtLevel; nxtLevel ← {}
    for u ∈ curLevel do
        for v ∈ N(u) do
            put vertex v into nxtLevel
        end for
    end for
    nxtLevel ← sortUnique(nxtLevel)
    nxtLevel ← nxtLevel − curLevel
end while
end procedure
```

Splitting level states





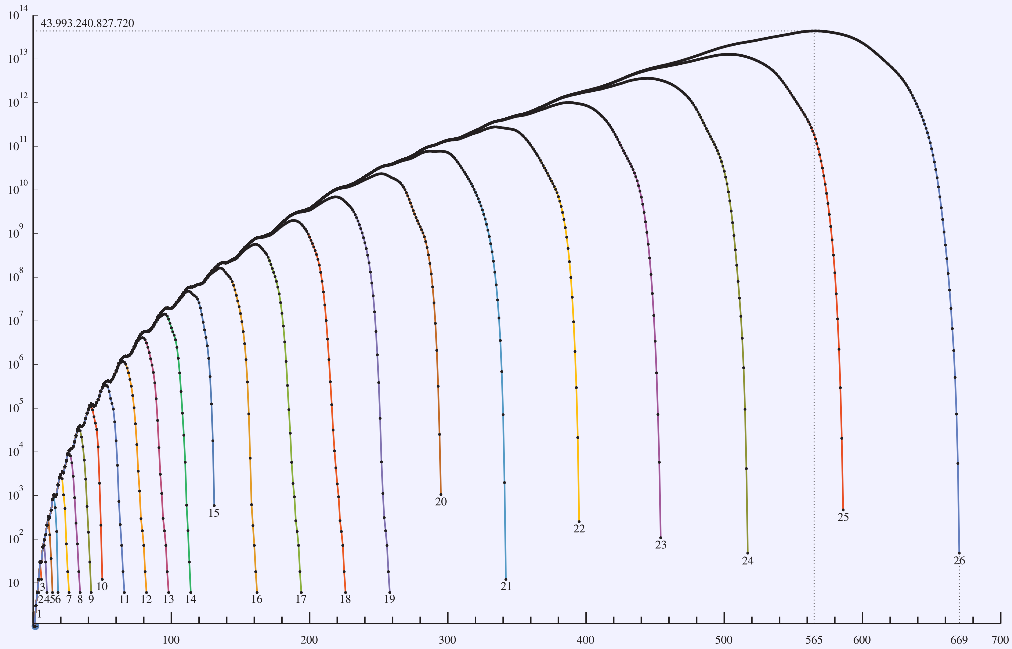
Generating next tree level



Some implementation details

- all file I/O operations are buffered (4 KB)
- we keep information about existence of all files also for programmatic later removal, since the cost of querying the file system is high
- workers are appending data concurrently into the same files, atomicity of operation fwrite is assurred on GPFS
- we have structured directories on the file system preventing too many files in the same directory
- since we have tasks of uneven size, we first sort them by size and then push them from largest to smallest using the "producer-consumer scheme"; this way the load per workers-consumers becomes quite even and consequently we have minimized wait time before `MPI_Barrier`

Graph growth from a perfect state in $H_4^n, n \leq 26$



To store the largest level 565 of the graph H_4^{26} we needed approx. 330 TB of space on GPFS. The algorithm needs two consecutive levels at the same time.

Equivalence of states

Two states are considered to be equivalent if one state emanates from the other by a permutation of the pegs:

$$s \sim s' :\Leftrightarrow \exists\, \sigma \in S_p : \sigma \circ s = s',$$

where $\sigma \circ s := \sigma(s_n) \ldots \sigma(s_1)$.

A formula for the number of equivalence classes (also called *equi-sets*) of states on H_p^n depending on p and n can be derived using Burnside's Lemma.

$$|V(H_p^n)/\sim| = \frac{1}{p!} \sum_{q=1}^{p} \binom{p}{q} q^n (p-q)\text{¡} = \sum_{q=0}^{p} q^n \frac{(p-q)\text{¡}}{q!(p-q)!}$$

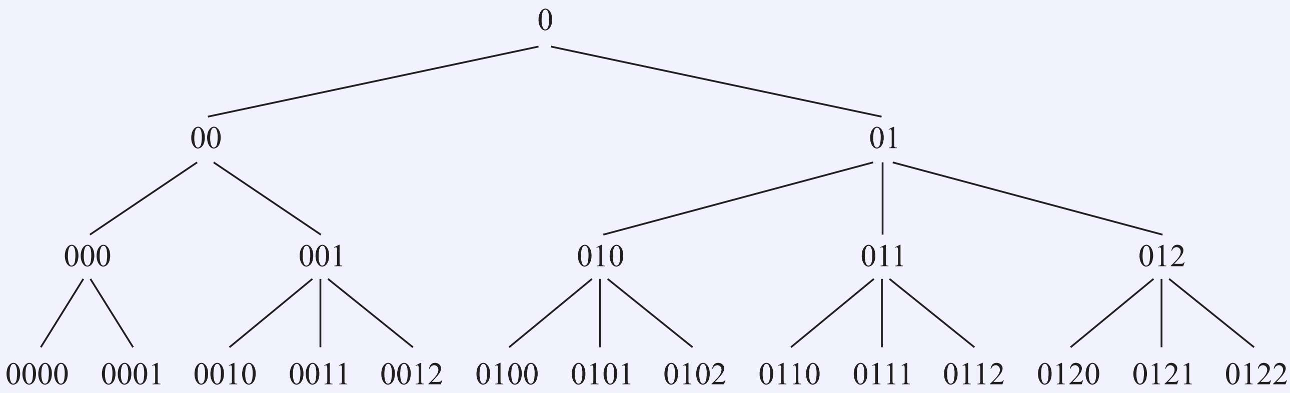
where

$$k\text{¡} := k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}$$

is the *subfactorial* of k, representing the number of derangements on $[k]$.

Generating representatives of equivalence classes of H_3^4

K.A.M. Götz (2008)



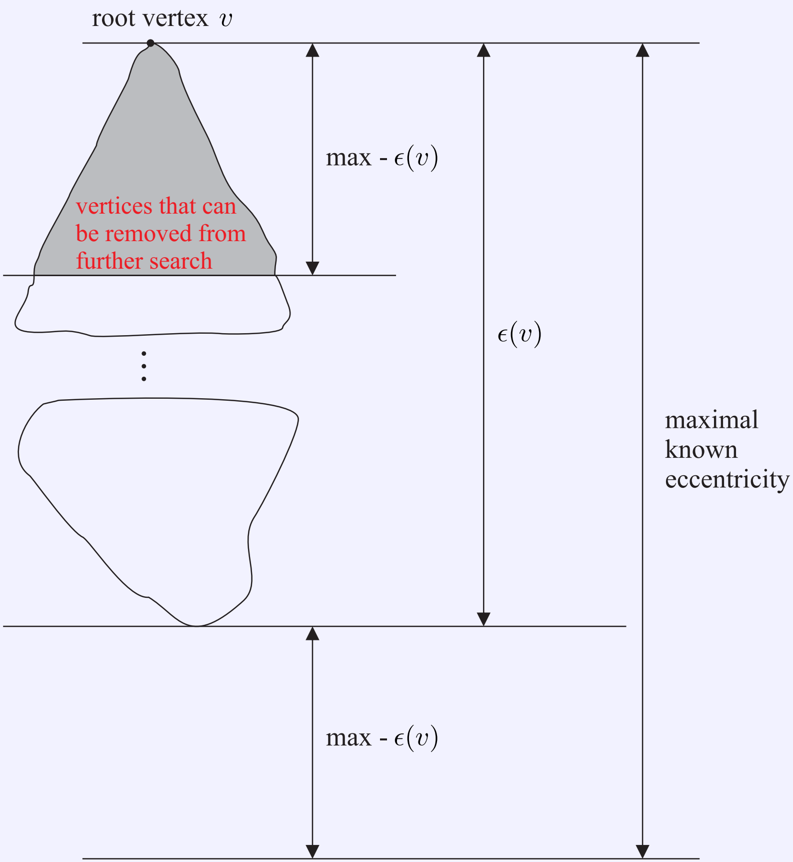
In each level a new disc is added to all already occupied pegs and to the first empty peg.

Computing $\mathrm{diam}(H_4^n)$

n	p^n	equi sets
1	4	1
2	16	2
3	64	5
4	256	15
5	1024	51
6	4096	187
7	16 384	715
8	65 536	2795
9	262 144	11 051
10	1 048 576	43 947
11	4 194 304	175 275
12	16 777 216	700 075
13	67 108 864	2 798 251
14	268 435 456	11 188 907
15	1 073 741 824	44 747 435
16	4 294 967 296	178 973 355
17	17 179 869 184	715 860 651
18	68 719 476 736	2 863 377 067
19	274 877 906 944	11 453 377 195
20	1 099 511 627 776	45 813 246 635

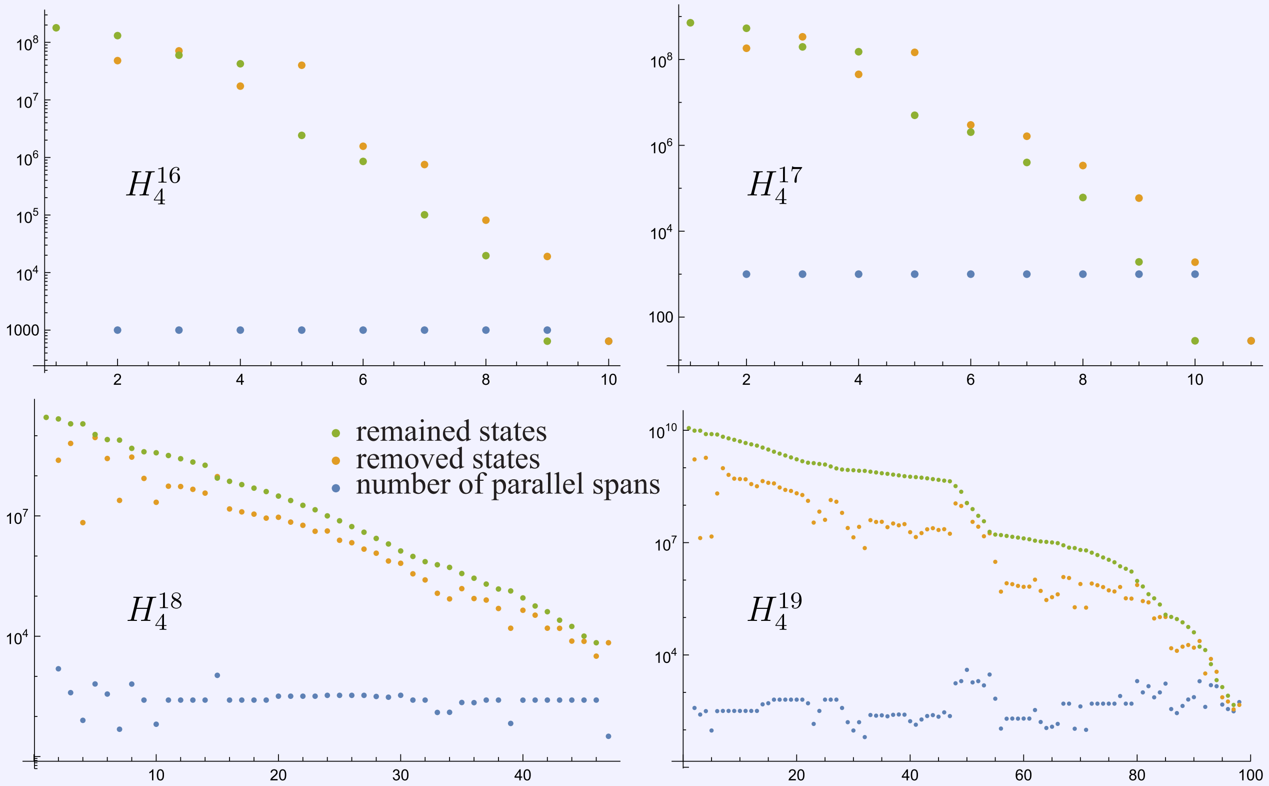
Since the diameter is the maximal eccentricity, one should compute the eccentricity (i.e. span a tree) for one representative of each equivalence class. Luckily we have a method to reduce the search space.

Reducing the search space



Reductions of the search space through sequential batches of spans in H_4^n using jobs farming.

We have executed (8642, 9028, 14332, 57880) spans grouped into (9, 10, 46, 97) batches.



2. Results and Outlook

Frame-Stewart conjecture has been confirmed for

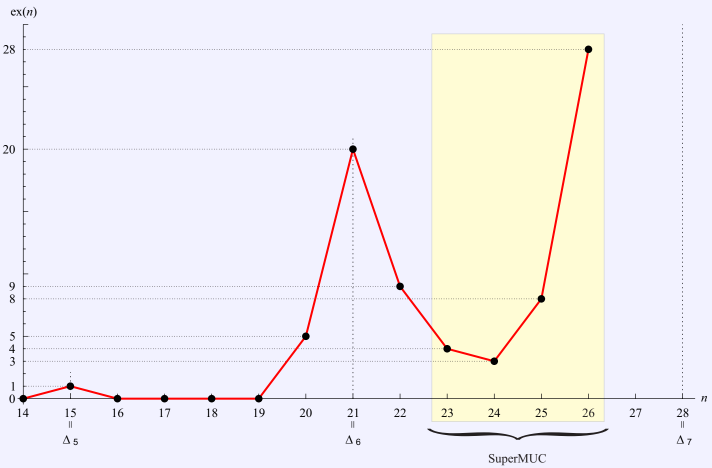
$$p = 5 \text{ and } n \leq 20,\ p = 6 \text{ and } n \leq 16,\ p = 7 \text{ and } n \leq 21.$$

Subtower conjecture has been confirmed for $p \leq 7$ and $n \leq \binom{p}{2}$.

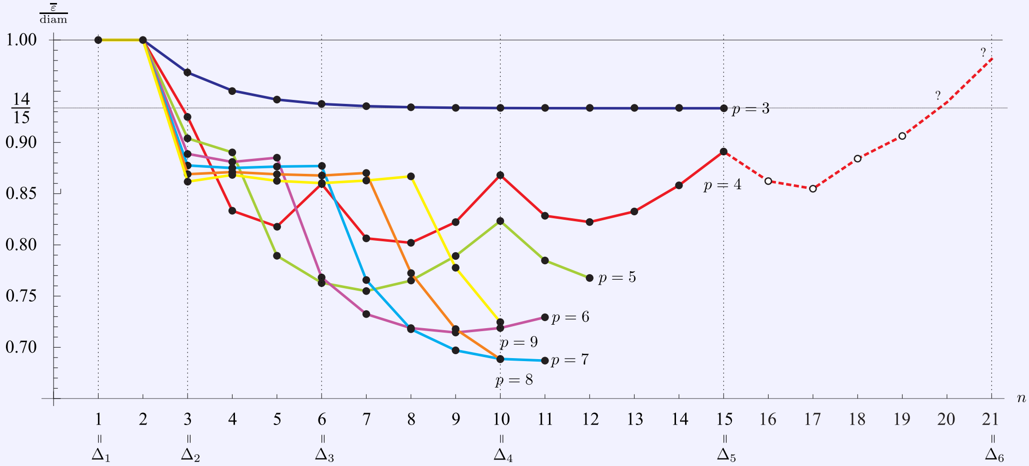
Korf-Felner conjecture has been confirmed for $n \leq 26$:

n	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$\mathrm{d}_4(0^n, 3^n)$	97	113	129	161	193	225	257	289	321	385	449	513	577	641
$\varepsilon_4(0^n)$	97	113	**130**	161	193	225	257	**294**	**341**	**394**	**453**	**516**	**585**	**669**
$\mathrm{ex}(n)$	0	0	**1**	0	0	0	0	**5**	**20**	**9**	**4**	**3**	**8**	**28**
$\mathrm{EX}(n)$	0	0	1	0	0	0	0	≥ 5	≥ 20	≥ 9	≥ 4	≥ 3	≥ 8	≥ 28

No Korf phenomenon detected for $p > 4$ and accessible n.



ex(n)
28
20
9
8
5
4
3
1
0
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
n
Δ 5
Δ 6
Δ 7
SuperMUC



$\frac{\overline{\varepsilon}}{\mathrm{diam}}$
1.00
$\frac{14}{15}$
0.90
0.85
0.80
0.75
0.70
$p = 3$
$p = 4$
$p = 5$
$p = 6$
$p = 9$
$p = 7$
$p = 8$
?
?
n
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
Δ_1
Δ_2
Δ_3
Δ_4
Δ_5
Δ_6

Variations of the Tower of Hanoi

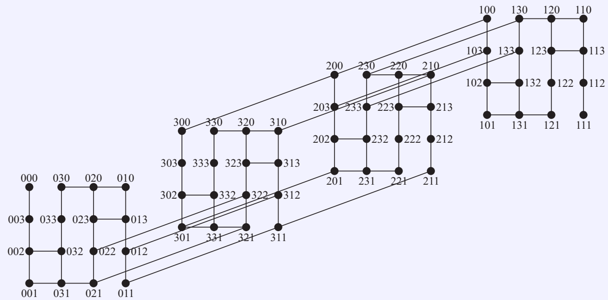
Star Tower of Hanoi $H^n_{K_{1,3}}$: A291877$(n) = \mathrm{d}_{K_{1,3}}(0^n, 1^n)$ (with B. Lužar)

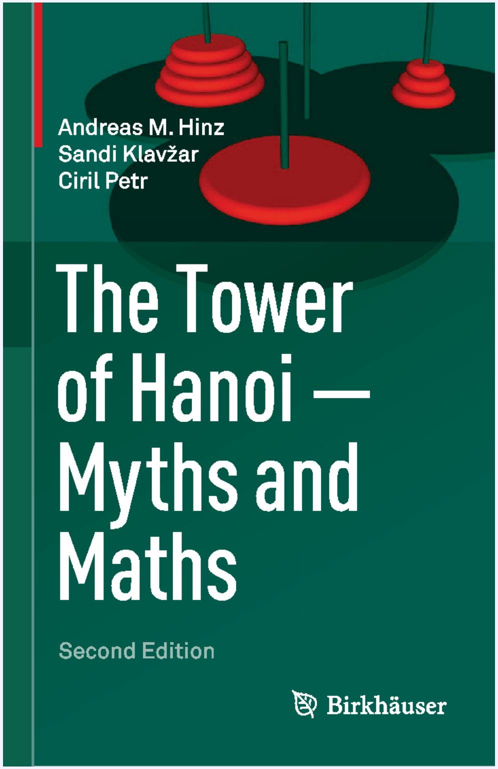
n	16	17	18	19	20	21	22
$\mathrm{d}_{K_{1,3}}(0^n, 1^n)$	480	579	700	835	1 012	1 201	1 428

Linear Tower of Hanoi $H^n_{P_{1+3}}$: A160002$(n) = \mathrm{d}_{P_{1+3}}(0^n, 1^n)$

n	21	22	23
$\mathrm{d}_{P_{1+3}}(0^n, 1^n)$	4 377	5 276	**6 247**

$H^3_{P_{1+3}}$





Cham, 2018

http://www.tohbook.info

Further reading:

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