Aptitude Test Automatic Control (SAMPLE) – SOLUTION

Examiner: Prof. Dr.-Ing. Florian Holzapfel
2022-12-24

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(to be filled by staff)

Exam
Editing Time: 1h

Version: unspecified
Good luck!
Consider two frictionless carts with the masses \(3M\) and \(M\). The left cart is connected to the left wall by a spring with the stiffness \(k_1\) and a damper with the damping coefficient \(c_1\), the right cart is connected to the right wall by a spring with the stiffness \(k_2\). The left cart is excited by an external force \(F\). Additionally, both carts are coupled with a spring with a stiffness of \(k_{1,2}\). In the equilibrium position \(p_1 = p_2 = 0\) all springs are tension-free. All springs and dampers are considered massless.

a) Use Newton’s second law to derive the equation of motion of the left cart in the form \(\ddot{p}_1 = f(p_1, p_2, \dot{p}_2, F)\).

\[
3M \ddot{p}_1 = F - k_1 p_1 - c_1 \dot{p}_1 + k_{1,2} (p_2 - p_1) \\
\dot{p}_1 = \frac{1}{3M} (- (k_1 + k_{1,2}) p_1 - c_1 \dot{p}_1 + k_{1,2} p_2 + F)
\]
b) Use Newton’s second law to derive the equation of motion of the right cart in the form 
\[ \ddot{p}_2 = f(p_1, p_1, p_2, \dot{p}_2, F). \]

\[ M \ddot{p}_2 = k_{1,2} (p_1 - p_2) - k_2 p_2 \]
\[ \dot{p}_2 = \frac{1}{M} (k_{1,2} p_1 - (k_2 + k_{1,2}) p_2) \]

\( (ca. 2 \text{ points}) \)

c) Derive a state space model of the form \( \dot{x} = A x + b u, \ y = c^T x \). The input \( u \) is the external force \( F \) and the distance \( p_2 - p_1 \) shall be considered as output signal \( y \). Use the state vector \( x = [ p_1, \dot{p}_1, p_2, \dot{p}_2 ]^T \).

\[ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{k_{1,2}}{M} & \frac{1}{M} & 0 \\ \frac{k_{1,2}}{M} & 0 & 0 & 0 \\ \frac{k_{1,2}}{M} & 0 & 0 & \frac{1}{M} \\ -\frac{k_{2} + k_{1,2}}{M} & 0 & -\frac{k_{2} + k_{1,2}}{M} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \dot{p}_1 \\ p_2 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F \]

\[ y = p_2 - p_1 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \dot{p}_1 \\ p_2 \\ \dot{p}_2 \end{bmatrix}^T \]

\( (ca. 3 \text{ points}) \)
Task 2: Laplace Transform

The differential equation
\[ \dot{y}(t) + \sigma(t) \cdot y(t) = 2 \cdot u(t) \]
of a system is given. The initial value is \( y(0) = \). Consider the following hints regarding notation and the given Laplace correspondences.

a) Determine the step response \( h(t) \) and the impulse response \( g(t) \) of the system.

\[ \dot{y}(t) + \sigma(t) \cdot y(t) = 2 \cdot u(t) \]

\[ sY(s) - y(0) + \frac{1}{s} Y(s) = 2U(s) \]

\[ \Rightarrow Y(s) = \frac{2s}{s^2 - 1} U(s) \]

\[ U(s) = \frac{1}{s} \Rightarrow Y(s) = H(s) = \frac{2}{s^2 + 1} \]

\[ h(t) = 2 \sin(t) \]

\[ g(t) = \dot{h}(t) = 2 \cos(t) \]
b) The system of differential equations is now given as
\[
\begin{align*}
\dot{y} &= -x + 2u \\
\dot{x} &= f(y, x, u)
\end{align*}
\]  
(1)

Compute \( f(x, y, u) \) such that the system of differential equations is equivalent to the system from a).

(ca. 2 points)
Task 3: Bode Diagram

The transfer function of a dynamic system can be represented by the series connection of the two transfer functions $G_1(s)$ and $G_2(s)$:

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) G_2(s)$$

The following diagram contains approximations of the magnitude and phase response curves for the element $G_1(s)$. The transfer function $G_2(s)$ is:

$$G_2(s) = \frac{20s + 4000}{(s + 20)^2}$$

Tasks: see next pages.
a) Determine the corner frequencies of $G_2(s)$ as well as approximations of the initial value $|G_2(\omega_{min})|$ and gradient $|G_2(\omega_{min})'|$ of the magnitude response in dB and dB/decade and the initial value of the phase response $\angle G_2(\omega_{min})$.

*(ca. 4 points)*

**Hint:** $\omega_{min} = 10^{-1}$

Corner frequencies:

$$G_2(s) = \frac{20s + 4000}{(s + 20)^2} = \frac{10s/200 + 1}{(s/20 + 1)^2}$$

$$\tilde{T}_1 = \frac{1}{200} \Rightarrow \tilde{\omega}_1 = \frac{1}{|\tilde{T}_1|} = 200$$

$$T_{1,2} = \frac{1}{20} \Rightarrow \omega_{1,2} = \frac{1}{|T_{1,2}|} = 20$$

Gain $K = 10$, no poles or zeros at 0:

$$\Rightarrow |G_2(\omega_{min})| \approx 20 \log_{10}(10) = 20 \text{ dB}$$

$$\Rightarrow |G_2(\omega_{min})'| \approx 0 \text{ dB/decade}$$

$$\Rightarrow \angle G_2(\omega_{min}) \approx 0^\circ$$

---

b) Draw the approximation of the magnitude and phase response curves of $G_2(s)$ in the diagram on the next page. *(ca. 4 points)*

c) Finally, draw the approximation of the magnitude and phase response curves of the entire system $G(s)$ in the diagram on the next page. *(ca. 2 points)*
Task 4: System Analysis / Steady-State Accuracy (ca. 3 points)

The following stable closed-loop system is given:

\[ \frac{2(s + 1)}{0.1s + 1} \quad \frac{s + 2}{s^2 + 4s + 3} \quad \frac{1}{s^2 + 2s} \]

\[ R(s) \quad G_1(s) \quad G_2(s) \]

a) Evaluate the steady-state accuracy of the closed-loop system with regard to the reference tracking behavior of the control loop (reference variable: \( w \)) assuming a step input in \( w \) and justify your statement.

\[(ca. 1 \text{ point})\]

Integration in \( R G_1 G_2 \), closed-loop stable \( \Rightarrow \) steady-state accurate reference tracking.

b) Evaluate the steady-state accuracy of the closed-loop system with regard to the disturbance behavior of the control loop (disturbance variable: \( z \)) assuming a step input in \( z \) and justify your statement.

\[(ca. 1 \text{ point})\]

No integration in \( R G_1 \) \( \Rightarrow \) disturbance behavior not steady-state accurate.

c) What type of the controller \( R(s) \) is used? If necessary, distinguish between ideal and real implementation.

\[(ca. 1 \text{ point})\]

Real PD controller.
Task 5: Input/Output Linearization (ca. 7 points)

Consider a dynamic system of the form $\dot{x} = f(x, u)$, $y = h(x)$ with the states $x = [x_1, x_2, x_3]^T$, input signal $u$ and output signal $y$:

$x \in \mathbb{R}^n$

$f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, $(x, u) \mapsto \dot{x}$

$u \in \mathbb{R}$

$h : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto y$

a) Describe in general terms the principle of input-output linearization for single input single output (SISO) systems of this form.

The system output $y$ is differentiated until an explicit influence by the input $u$ appears at derivative order $r$ (the relative degree). Solving the relation between $\frac{\partial^r y}{\partial u}$ and $u$ for the input signal $u$ yields a control law that compensates the nonlinearity of the system by inversion such that linear input-output dynamics result, with the new input $\frac{\partial^r y}{\partial u}$ acting as pseudo-control.

In the following the system $S$ is considered:

$$S : f(x, u) = \begin{bmatrix} \sin(x_2) + (x_2 + 1)x_3 \\ x_1^5 + x_3 \\ x_1^2 + u \end{bmatrix}, \quad h(x) = x_1$$
b) Differentiate the output of the system $S$ up to the relative degree and explicitly state the relation to the input signal.

\[(ca. 2 \text{ points})\]

\[
\begin{align*}
y &= x_1 \\
\dot{y} &= \dot{x}_1 = \sin(x_2) + (x_2 + 1) x_3 \\
\ddot{y} &= \left(\cos(x_2) + x_3\right) \dot{x}_2 + (x_2 + 1) \dot{x}_3 \\
\dddot{y} &= \left(\cos(x_2) + x_3\right) \left(x_5^2 + x_3\right) + (x_2 + 1) \left(x_1^2 + u\right)
\end{align*}
\]

c) State a pseudo-control variable $\nu$ for the system $S$.

\[(ca. 1 \text{ point})\]

\[
\begin{align*}
\nu &= \ddot{y}
\end{align*}
\]

d) Calculate a control law $u = g(x, \nu)$ for the system $S$ that linearizes the dynamics between $\nu$ and $y$.

\[(ca. 2 \text{ points})\]

\[
\begin{align*}
u &= \frac{\nu - \left(\cos(x_2) + x_3\right) \left(x_5^2 + x_3\right) - (x_2 + 1) x_1^2}{(x_2 + 1)}
\end{align*}
\]
Task 6: Nyquist Criterion (ca. 3 points)

The following plot shows the open-loop frequency response $G(j\omega)$ of the open-loop system $G(s)$ for all relevant frequencies $\omega > 0$. The closed-loop stability shall be assessed using the general Nyquist criterion.

- Calculate the argument change $W^*_+$ of $G(j\omega) + 1$ required for closed-loop stability.
- Determine the actual argument change $W_+$.
- Determine the stability of the closed-loop system.

\[
G(s) = 220 \exp(-s) \frac{(s + 0.05)(s - 1)}{s(s + 10)^2(s - 5)}
\]

- one neutral and one unstable pole
  \[ W^*_+ = \frac{1}{2} \pi + 1\pi = \frac{3\pi}{2} \] (1P)
- from locus: $W_+ = -\frac{3\pi}{2}$ (1P)
- $W_+ \neq W^*_+$ $\Rightarrow$ closed-loop system unstable (1P)

- alternative solution:
  \[ \omega \in (-\infty, \infty) \Rightarrow W^*_+ = 3\pi \neq -3\pi = W_+ \]
Task 7: State Observer (ca. 5 points)

The system $\Sigma$ is given. To estimate its state $x$, a state observer $L$ is used. The estimation error $\hat{x}$ is defined as $\hat{x} := x - \hat{x}$.

$$
\begin{align*}
\Sigma : & \quad \dot{x} = Ax \\
L : & \quad \dot{\hat{x}} = A\hat{x} - k (y - \hat{y}) \\

\end{align*}
$$

a) Derive the differential equation $\dot{\hat{x}} = \hat{A} \hat{x}$ for the estimation error dynamics. (ca. 2 points)

\[
\dot{\hat{x}} := x - \hat{x} \implies \dot{\hat{x}} = \dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + k c^T (x - \hat{x}) = \underbrace{(A + k c^T)\hat{x}}_\hat{A}
\]
Now the following values are given:

\[ A = \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix}, \quad k = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad c^T = [1 \ 2] \]

b) Calculate the matrix \( \tilde{A} \) and its eigenvalues. 

\[
\tilde{A} = (A + k c^T)
= \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}
= \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}
\]

\[ \Rightarrow \lambda_1 = -3, \quad \lambda_2 = -2 \]

(c) Does the estimation error vanish for \( t \to \infty \)? Justify your answer.

\[
\text{Re}(\lambda_{1,2}) < 0 \quad \Rightarrow \quad \lim_{t \to \infty} \dot{x} = 0
\]
Task 8: Transfer Function

Calculate the transfer function $G(s) = \frac{X(s)}{F(s)}$ for the system of second order differential equations

$$M \ddot{q} = -R \dot{q} + e f$$

with the output equation $x = a^T q$.

Hint: $M \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times n}$, $q(t) \in \mathbb{R}^{n \times 1}$, $e \in \mathbb{R}^{n \times 1}$, $a^T \in \mathbb{R}^{1 \times n}$, $f(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}$

\[
\begin{align*}
M s^2 Q(s) + R s Q(s) &= e F(s) \quad (1P) \\
(M s^2 + R s) Q(s) &= e F(s) \\
Q(s) &= (M s^2 + R s)^{-1} e F(s) \quad (1P) \\
X(s) &= a^T Q(s) \\
X(s) &= a^T \underbrace{(M s^2 + R s)^{-1}}_{G(s)} e F(s) \quad (1P)
\end{align*}
\]
Task 9: Control Design / State Feedback

Consider the following system with state vector $x$, control input $u$ and output signal $y$:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 \\ 0 \end{bmatrix} e^x$$

a) Calculate the feedback gain $r$ of the following control law such that the closed-loop eigenvalues become $\lambda_{R,1}, \lambda_{R,2}$.

$$u = -r^T x, \quad r \in \mathbb{R}^2, \quad \lambda_{R,1} = -2, \quad \lambda_{R,2} = -3$$

\begin{align*}
\mathbf{r} &= [r_1, r_2]^T \\
\mathbf{A}_R &= \mathbf{A} - \mathbf{b}r^T \\
(sI - \mathbf{A}_R) &= \begin{bmatrix} s + 1 & -1 \\ 1 - r_1 & s - r_2 + 1 \end{bmatrix} \\
\det(sI - \mathbf{A}_R) &= s^2 + (2 - r_2)s + (2 - r_2 - r_1) \\
\prod_k (s - \lambda_{R,k}) &= s^2 + 5s + 6 \\
(2 - r_2) &= 5 \quad \Rightarrow \quad r_2 = -3 \\
(2 - r_2 - r_1) &= 6 \quad \Rightarrow \quad r_1 = -1 \\
\Rightarrow \mathbf{r} &= [-1, -3]^T
\end{align*}
b) The output signal $y$ shall track the reference signal $w$ with steady-state accuracy. Calculate the corresponding parameter $m_u$ of the following control law.

$$ u = -r^T x + m_u w, \quad m_u \in \mathbb{R}, \quad r = [-4, -2]^T $$

(ca. 4 points)

\[
\begin{align*}
A_w &= A - br^T \\
b_w &= bm_u \\
(sI - A_w) &= \begin{bmatrix} s + 1 & -1 \\ 5 & s + 3 \end{bmatrix} \quad \text{(1P)} \\
(sI - A_w)^{-1} &= \frac{1}{s^2 + 4s + 8} \begin{bmatrix} s + 3 & 1 \\ -5 & s + 1 \end{bmatrix} \quad \text{(1P)} \\
G_{yw}(s) &= c^T (sI - A_w)^{-1} b_w = \frac{-5 m_u}{s^2 + 4s + 8} \quad \text{(1P)} \\
limit_{s \to 0} G_{yw}(s) &= \frac{-5}{8} m_u \quad (0.5P) \\
\Rightarrow m_u &= \frac{-8}{5} \quad (0.5P)
\end{align*}
\]
Task 10: Control Structures / Disturbance Rejection

Consider the following system with reference signal $w$, control input $u$, output signal $y$, plant $G_1(s) \ldots G_4(s)$ and controller $R(s)$. The system is affected by the disturbance $z$, which is measured. A disturbance rejection filter $F_A(s)$ shall be designed.

a) Calculate the open-loop transfer functions $G_{yu}(s)$ and $G_{yz}(s)$ from the control input and disturbance to the output. Here, $F_A(s) = 0$. (ca. 2 points)

Auxiliary cut $v$ at the input to $G_3$:

$$v = (1 + G_2)z + G_1 G_2 u - G_2 G_4 (1 + G_3) v$$

$$\Rightarrow \quad G_{vu}(s)|_{z=0} = \frac{G_1 G_2}{1 + G_2 G_4 (1 + G_3)}$$

$$\Rightarrow \quad G_{vz}(s)|_{u=0} = \frac{1 + G_2}{1 + G_2 G_4 (1 + G_3)}$$

$$G_{yu} = G_3$$

$$\Rightarrow \quad G_{yu}(s) = \frac{G_1 G_2 G_3}{1 + G_2 G_4 (1 + G_3)} \quad \text{(1P)}$$

$$\Rightarrow \quad G_{yz}(s) = \frac{(1 + G_2) G_3}{1 + G_2 G_4 (1 + G_3)} \quad \text{(1P)}$$
b) Using the following transfer functions, calculate an appropriate disturbance rejection filter \( F_A(s) \) that can be implemented in practice. If necessary, apply a filter of the form \( V(s) \) (give reasons for this!).

*Note: This task can be solved independently.*

\[
G_{yu} = \frac{s - 2}{(s + 2)(3s^2 + 13s - 8)} \quad G_{yz} = \frac{2s + 1}{3s^2 + 13s - 8}
\]

\[
V(s) = \frac{(s - a)^m}{(Ts + 1)^n} \quad T = 0.01, \ a \in \mathbb{R}, \ m, n \in \mathbb{Z}_{\geq 0}
\]

(\( ca. 3 \) points)

\[
u = F_A z \\
y = G_{yu} u + G_{yz} z = (G_{yu} F_A + G_{yz}) z \neq 0 \ \forall z
\]

\[\Rightarrow \quad F_{A,\text{ideal}} = -G_{yu}^{-1} G_{yz} = \frac{(s + 2)(2s + 1)}{s - 2}
\] (1P)

\( F_{A,\text{ideal}} \) is unstable and violates causality. We need to cancel the unstable pole \( s_1 = 2 \) with a right half-plane zero and use a second-order low-pass filter to restore causality. (1P)

\[
F_A = \frac{s - 2}{(s + 1)^2} \quad F_{A,\text{ideal}} = \frac{(s + 2)(2s + 1)}{(Ts + 1)^2} = 2 \left( \frac{s + 2}{(0.01s + 1)^2} \right)
\] (1P)
Task 11: Discrete-Time Control

For the implementation of a discrete time filter on a digital micro controller the continuous time transfer function $G(s)$ shall be discretized and the difference equation of the discrete time filter shall be derived.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{20}{(s + 10)}$$

a) Derive the discrete time transfer function $R(z) = \frac{Y(z)}{U(z)}$. The sampling frequency of the discrete time filter is $f_T = 5$ Hz. Use the following transformation:

$$s = \frac{1}{T}(1 - z^{-1})$$

$$R(z) = \frac{20}{\frac{T}{1 - z^{-1}} + 10} = \frac{20T}{(1 - z^{-1}) + 10T}$$

$$R(z) = \frac{4}{\frac{3}{1 - z^{-1}} - 1} = \frac{4z}{3z - 1}$$
b) Derive the difference equation for the computation of the filter output signal in the following form:

\[ y[k] = f(u[k], u[k-1], \ldots, y[k-1], \ldots) . \]

\[ R(z) = \frac{4}{3 - z^{-1}} \]

\[ 3 \cdot y[k] - z^{-1} y[k] = 4 \cdot u[k] \]

\[ y[k] = \frac{1}{3}(y[k-1] + 4 \cdot u[k]) \]

(c. 1 point)

c) At the time step \( k = 1 \), a unit step signal is applied at the filter input: \( u[k > 0] = 1 \). The initial output value of the filter is \( y[k = 0] = 4 \). What is the value of the filter output signal \( y[k] \) at the time step \( k = 2 \)?

\[ y[0] = 4 \]

\[ y[1] = \frac{1}{3}(4 + 4 \cdot 1) = \frac{8}{3} \]

\[ y[2] = \frac{1}{3}(\frac{8}{3} + 4 \cdot 1) = \frac{20}{9} \]

(c. 1 point)
Task 12: Short Questions  

tick the correct answer.

a) Which of the following elements leads to steady-state accuracy?  

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<tr>
<td>Differentiator</td>
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<tr>
<td>Low pass filter</td>
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<td>High pass filter</td>
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b) The error signal of a standard feedback control system depends on:  

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<tr>
<td>Command and plant input signal</td>
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<tr>
<td>Plant input and output variable.</td>
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<tr>
<td>Model parameters.</td>
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c) An open-loop control system has the following property:

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<tr>
<td>The input variables are independent of the output signals.</td>
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<tr>
<td>The input variables depend on the size of the system.</td>
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<tr>
<td>The input variables depend on plant parameters.</td>
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(ca. 1 point)
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