Chair of Astronautics
Department of Aerospace and Geodesy
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Esolution
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Mathematics

Exam: LRXXXX / Qualification exam
Examiner: Prof. Dr. Ulrich Walter
Date: Sunday 31st February, 2999
Time: 10:00 – 11:00

	P 1	P 2	P 3	P 4	P 5	P 6	P 7	P 8	P 9	P 10	P 11	P 12	P 13
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Working instructions

- This exam consists of **13 pages** with a total of **13 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **non-programmable pocket calculator**
  - one **analog dictionary** English ↔ native language
- **Answers are only accepted if the solution approach is documented.** Give reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices including smart watches, put them into your bag and close the bag.
- In multile choice problems, only one answer is correct.
- Additional space for solutions is provided at the back of the exam. If required, clearly state that you used the addional space in the problem's solution box and reference the corresponding problem with the solution.

Left room from __________ to __________ / Early submission at __________

- Intentionally left blank -

Problem 1 **Short Questions (8 credits)**

Select the solution of the following short questions without providing the approach. There is only one correct answer per problem.
One point for each correct answer ①.

a) Solve $e^{\frac{5\pi}{2}i}$.

- [ ] $-1 + 1i$
- [ ] $0 - i$
- [ ] $1 + 0i$
- [x] $0 + i$

b) The limit of the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

- [ ] 0
- [ ] ∞
- [x] 2
- [ ] π

c) Roots of $x^3 + 4x, x \in \mathbb{C}$.

- [ ] $[2i]$
- [x] $[0, -2i, 2i]$
- [ ] $[0, 2, -2]$
- [ ] $[0]$

d) Solution of $\frac{x^2+1}{\sin x} = 0, x \in \mathbb{R}$.

- [ ] $[0]$
- [ ] $[i, -i]$
- [ ] $[-2, 0, 2]$
- [x] $\emptyset$

e) The inverse E^{-1} of $E = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$.

- [ ] $\begin{bmatrix} 1.5 & -0.5 \\ 1 & 0 \end{bmatrix}$
- [x] $\begin{bmatrix} -1.5 & 0.5 \\ 1 & 0 \end{bmatrix}$
- [ ] $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
- [ ] $\begin{bmatrix} 3 & 1 \\ 0.5 & 1 \end{bmatrix}$

f) The determinate $\det(F)$ of the matrix $F = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$.

- [ ] 42
- [ ] 17
- [ ] -4
- [x] 0

g) The limit $\lim_{x\to 2+} \frac{x^2-4}{x-2}$.

- [ ] ∞
- [x] 4
- [ ] $-\infty$
- [ ] 0

h) Select a possible Fourier series represenation of a sinusoidal alternate current with 0 V – 230 V and 60 Hz.

- [ ] $0 - 230 \exp(120 i\pi t)$
- [x] $115 + 115 \exp(120 i\pi t)$
- [ ] $115 - 230 \exp(60 i\pi t)$
- [ ] $115 - 115 \exp(60 i\pi t)$

0 1 2 3 4 5 6 7 8

Problem 2 Matrix operations (4 credits)

0
1
2
3
4

Form - if possible - with the matrices

$$A = \begin{pmatrix} -2 & 3 \\ 4 & 1 \\ -1 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 1 & -7 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 4 \\ 0 & -2 \\ 3 & 5 \end{pmatrix}$$

and vectors

$$x = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, y = \begin{pmatrix} 8 \\ -5 \end{pmatrix} \text{ and } z = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

the expressions

$$A + C, \qquad 2B, \qquad A(y + z), \qquad C(-4z).$$

$$A + C = \begin{pmatrix} -1 & 7 \\ 4 & -1 \\ 2 & 10 \end{pmatrix} ① \qquad 2B = \begin{pmatrix} 6 & 0 \\ 2 & -14 \end{pmatrix} ① \qquad A(x + y) = Ax + Ay = \begin{pmatrix} -31 \\ 41 \\ -26 \end{pmatrix} ①$$

$$C(-4z) = -4Cz = \begin{pmatrix} -44 \\ 16 \\ -76 \end{pmatrix} ①$$

Problem 3 Scalar and vector product (5 credits)

Let $a = (1,0,2)^\top$ and $b = (1,2,0)^\top$ be two vectors of $\mathbb{R}^3$. Determine

a) the vector product c and the scalar product d of the vectors a and b.

Vector product: $c = a \times b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$ ①

Scalar product: $d = \langle a, b \rangle = a_1b_1 + a_2b_2 + a_3b_3 = 1$ ①

b) the cosine of the angle α between the vectors a and b.

$\cos\alpha = \frac{\|a \cdot b\|}{\|a\|\ \|b\|} = \frac{\|d\|}{\sqrt{5}\ \sqrt{5}} = \frac{1}{5}$ ①

c) the area A of the parallelogram between the two vectors.

The area is the L_2-norm of the vector product.

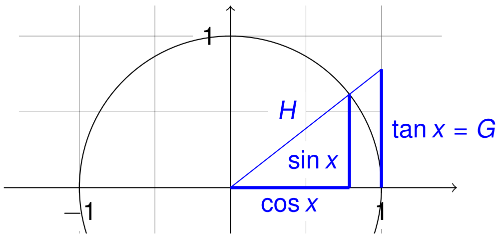
$A = \|a \times b\|_2 = \sqrt{24}$ ①

d) the unit normal vector n of a and b.

$n = \frac{1}{\sqrt{24}} \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$ ①

Problem 4 Trigonometry (3 credits)

a) Draw the sine, cosine and tangent of a arbitrary value x into the diagram below.



One point for the correct diagram. ①

b) Show that the following equation holds true using the Pythagorean theorem.

$$\sin x = \frac{\tan x}{\sqrt{1 + \tan^2 x}}$$

The Sine is defined as the quotient of the opposite G over the hypotenuse H

The Pythagorean theorem for the unit circle yields as hypotenuse

$H = \sqrt{1 + \tan^2 x}$ ①

Thus holds true for the sine:

$\sin x = \frac{G}{H} = \frac{\tan x}{\sqrt{1+\tan^2 x}}$ ①

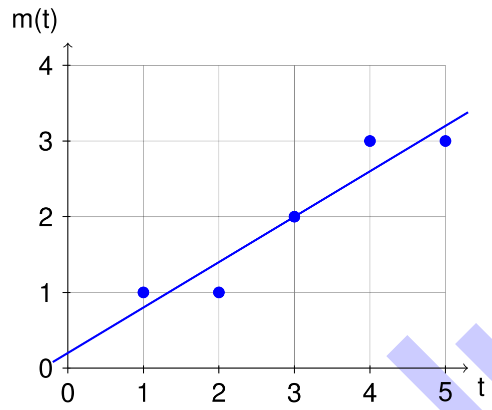
Problem 5 Linear regression (4 credits)

0 1 2 3 4

Let

t	1	2	3	4	5
m(t)	1	1	2	3	3

be a time series of measurements. Calculate the linear regression expression. Plot points and function in the provided diagram.



Searched is the equation $m(t) = x_0 + x_1 t$, therefore the solution of the linear system: $Ax = y$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} \text{①}$$

Solving the linear system: $A^\top Ax = A^\top y \leftrightarrow \begin{pmatrix} 5 & 15 \\ 15 & 55 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 36 \end{pmatrix} \leftrightarrow x = \begin{pmatrix} 1/5 \\ 3/5 \end{pmatrix}$ ①

Thus: $m(t) = \frac{1}{5} + \frac{3}{5}t$①

One point for the correct graphic.①

Problem 6 Limits (2 credits)

0 Determine the following limits.

1 (a) $\lim_{x\to 0} \frac{e^x - 1}{x}$

2 (b) $\lim_{x\to\infty} 2x - \sqrt{4x^2 - x}$

L'Hopital $\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$ ①

$2x - \sqrt{4x^2 - x} = \frac{4x^2 - 4x^2 + x}{x\left(2+\sqrt{4-\frac{1}{x}}\right)} = \frac{1}{2+\sqrt{4-\frac{1}{x}}} \xrightarrow[x\to\infty]{} \frac{1}{4}$ ①

Problem 7 Transformations (2 credits)

Select the solution of the following short questions without providing the approach. There is only one correct answer per problem.
One point for each correct answer ①.

0 The transformation $f : \mathbb{R} \to \mathbb{R}, n \mapsto n$ is

1 ☐ only surjective.

2 ☐ only injective.

☐ neither injective nor surjective.

☒ bijective.

The transformation $g : \mathbb{R} \to \mathbb{R}, n \mapsto n^2$ is

☐ only surjective.

☐ only injective.

☒ neither injective nor surjective.

☐ bijective.

Problem 8 Differentiation (8 credits)

a) Calculate the first derivative of the following functions.

$$f(x) = \ln(\sin x) - x\cos(x)$$
$$g(x) = -8\left(x + \frac{2}{x}\right) + 4\ln(x+3)$$
$$h(x) = x^2 \tan(x)$$

$f'(x) = \cot(x) - \cos(x) + x\sin(x)$①

$g'(x) = -8 + \frac{16}{x^2} + \frac{4}{x+3}$①

$h'(x) = 2x\tan(x) + \left(\frac{x}{\cos(x)}\right)^2$ ①

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^3 + y^3 - 3xy$$

Determine all local extrema of f.

Determination of gradient f: $\nabla f(x, y) = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix}$ ①

And Hessian matrix $H_f(x, y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ ①

Critical points are roots of the gradient: $\nabla f(x, y) = 0 \Leftrightarrow \begin{matrix} y = x^2 \\ x = 0 \\ x = 1 \end{matrix}$ gives $(0, 0)$ and $(1, 1)$.①

Point $(0, 0)$ in Hessian matrix: Negative determinant -> saddle point①

Point $(1, 1)$ in Hessian matrix: Positive determinant and positive trace -> local minimum①

Problem 9 Integration (4 credits)

0 1 2 3 4

Calculate the following specific and unspecific integral.

(a) $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

(b) $\int \frac{\ln(x^2)}{x^2}\,dx$

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx = \begin{bmatrix} u = 1+e^x \\ du = e^x dx \end{bmatrix} ① = \int_1^{1+e} \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_1^{1+e} = \frac{1}{2} - \frac{1}{1+e} ①$$

$$\int \frac{\ln(x^2)}{x^2}\,dx = \begin{bmatrix} u = \ln(x^2) & u' = \frac{2}{x} \\ v' = \frac{1}{x^2} & v = -\frac{1}{x} \end{bmatrix} ① = -\frac{\ln(x^2)}{x} + 2\int \frac{1}{x^2} + C = -\frac{\ln(x^2)}{x} + \frac{2}{x} + C ①$$

Problem 10 Series (1 credit)

0 1

Examine convergence of the following series and determine the limit, if applicable.

$$a_n = \frac{(4n+3)(n-2)}{n^2+n-2}$$

Converges, with $a_n \xrightarrow[n\to\infty]{} 4$ ①

Problem 11 Fourier series (5 credits)

Determine the Fourier series $F(x)$ of the 2π-periodic function $f : [-\pi, \pi) \to \mathbb{R}$ with

$$f(x) = \frac{1}{2}x^2$$

Tip: Be aware that the function $f(x)$ is even, i.e. axisymmetric.

The general forumla for a Fourier series is $F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{2\pi}{T}x\right) + b_k \sin\left(k\frac{2\pi}{T}x\right)$.①

$f(x)$ is even, thus $b_k = 0$ and $a_k = \frac{4}{T}\int_0^{T/2} f(x)\cos\left(k\frac{T}{2\pi}x\right)dx$.①

One obtains $a_0 = \frac{4}{2\pi}\int_0^{\pi}\frac{x^2}{2}dx = \frac{\pi^2}{3}$①

And $a_k = \frac{4}{2\pi}\int_0^{\pi}\frac{x^2}{2}\cos(kx)dx = \frac{2}{k^2}(-1)^k$.①

The Fourier series of f is then $F(x) = \frac{\pi^2}{6} + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx$.①

Problem 12 Taylor series (4 credits)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = 2x^3 + 3y^2 + \sin(x)\cos(y).$$

Determine the second order Taylor polynomial $T_{2,f,a}(x, y)$ at point $a = (0, 0)^\top$.

Taylor polynomial: $T_{2,f,a}(x) = f(a) + \nabla f(a)^\top(x-a) + \frac{1}{2}(x-a)^\top Hf(a)(x-a)$

Partial derivative up to 2. order: $\nabla f(x, y) = \begin{pmatrix} 6x^2 + \cos x\cos y \\ 6y - \sin x\sin y \end{pmatrix}$①

$Hf(x, y) = \begin{pmatrix} 12x - \sin x\cos y & -\cos x\sin y \\ -\cos x\sin y & 6 - \sin x\cos y \end{pmatrix}$①

Evaluation at point a: $f(a) = 0$, $\quad \nabla f(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad Hf(a) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$①

Yielding $T_{2,f,a}(x) = 0 + (1, 0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x, y)\begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x + 3y$①

Problem 13 Differential equations (10 credits)

0 1 2 3

a) Solve the following initial value problem by separation of variables.

$$\dot{x} = e^{x} \sin(t) \text{ with } x(0) = 0$$

Integration after separation of the variables $\int e^{-x}dx = \int \sin(t)dt \Leftrightarrow -e^{-x} = -\cos(t) + c$.①

Solving results in $x(t) = -\ln(\cos(t) + c)$.①

Inserting the initial condition yields $c = 0$, thus $x(t) = -\ln(\cos(t))$.①

0 1 2 3 4 5 6 7

b) Provide a general solution to the following differential equation

$$\ddot{x} - 7\dot{x} + 6x = \sin(t)$$

The solution is the sum of homogeneous and particular solution: $x(t) = x_h(t) + x_p(t)$.①

Solution of characteristic polynomial $\lambda^2 - 7\lambda + 6 = 0$ yields $\lambda_1 = 1$ and $\lambda_2 = 6$.①

The general solution of the homogeneous ODE is then $x_h(t) = c_1 e^{t} + c_2 e^{6t}$.①

The particulate solution can be solved with the approach $\begin{aligned} x_p(t) &= A\cos t + B\sin t \\ \dot{x}_p(t) &= -A\sin t + B\cos t \\ \ddot{x}_p(t) &= -A\cos t - B\sin t \end{aligned}$.①

Insertion in the ODE results in $(-A - 7B + 6A)\cos t + (-B + 7A + 6B)\sin t = \sin t$.①

A comparison of coefficients provides $A = \frac{7}{74}$ $B = \frac{5}{74}$.①

The general solution is then $x(t) = c_1 e^{t} + c_2 e^{6t} + \frac{7}{74}\cos t + \frac{5}{74}\sin t$.①

Additional solution space (use as required).