ON THE OBSERVABILITY OF STATE SPACE MODELS WITH GAUSSIAN DISTURBANCES

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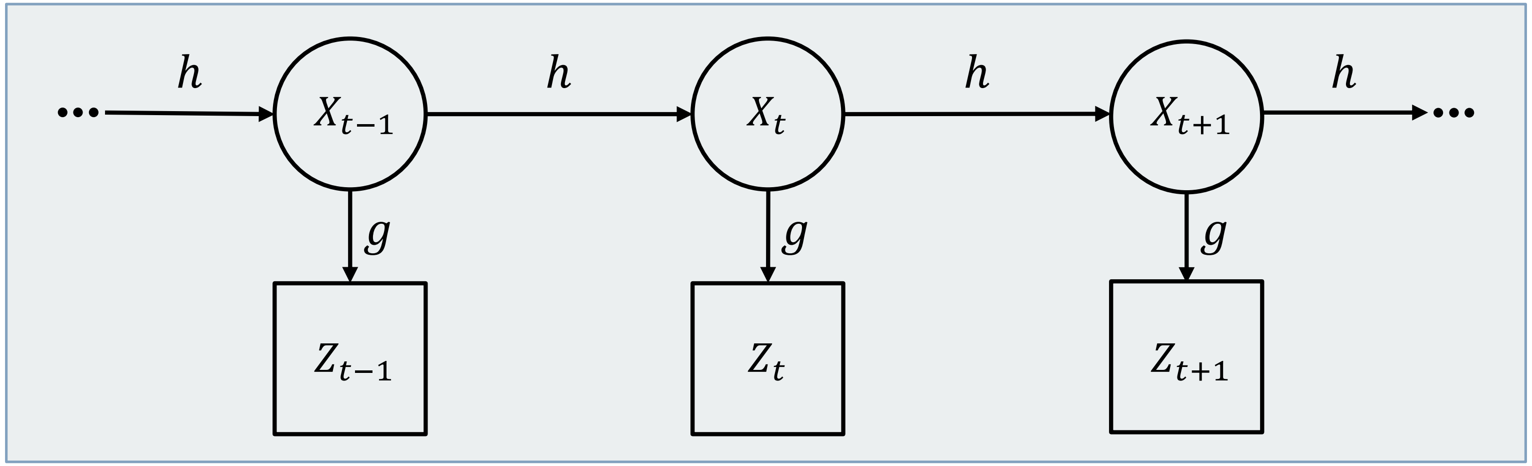
INTRODUCTION (I)



General State Space Models

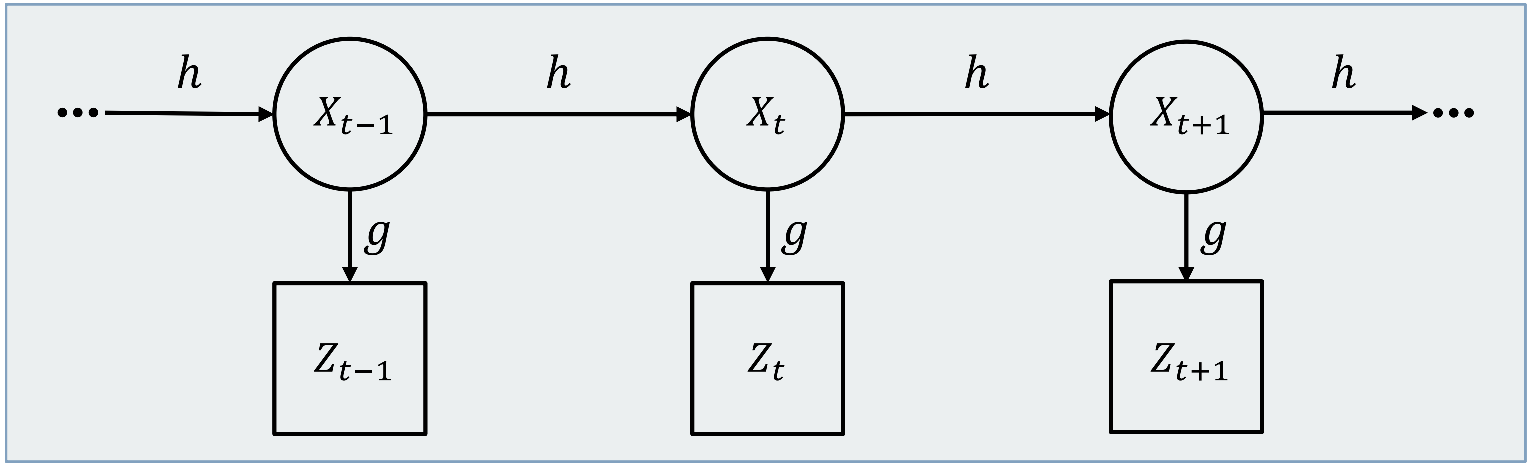
Observation at time t

$$\text{Observation equation: } Z_t = g(\mathbf{\Omega}, X_t, \epsilon_t)$$
$$\text{State equation: } X_t = h(\mathbf{\Omega}, X_{t-1}, \eta_t) \text{ for } t = 1, \ldots, T$$

Latent variable/State at time t

- $\mathbf{\Omega}$: vector of time-independent **parameters**
- $(\epsilon_t)_{t=1,\ldots,T}$ and $(\eta_t)_{t=1,\ldots,T}$ independent i.i.d. standard normal **disturbances**
- X_0 **inital state**, assume fixed value x_0^*

INTRODUCTION (I)



General State Space Models

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Latent variable/State at time t

Question: Observable?

Given observation vector $\mathbf{z}_\mathrm{T} = (z_1, \ldots, z_T)$ → Possible to estimate the unknown parameters?

→ We need some definitions

DEFINITIONS (1)

- Given: Realization $\boldsymbol{z}_{\mathrm{T}} = (z_1, \dots, z_T)$ of $\mathbf{Z}_{\mathrm{T}} = (Z_1, \dots, Z_T)$

- Underlying parameters we want to estimate:

Fixed

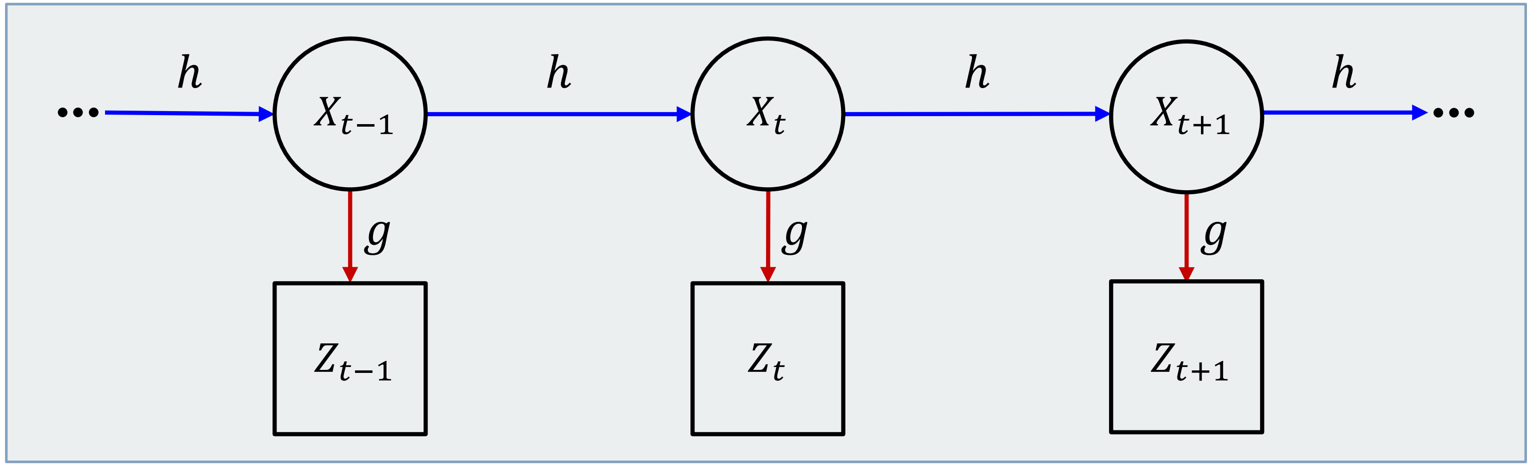
Different for every $\boldsymbol{z}_{\mathrm{T}}$

$$\boldsymbol{\Theta} = (\underbrace{\boldsymbol{\Omega}, x_0}, \mathbf{x}_T), \mathbf{x}_T = (x_1, \dots, x_T)$$

Parameters of interest

Nuisance parameters

DEFINITIONS (2)



Given: $\boldsymbol{z}_\mathrm{T} = (z_1, \dots, z_T)$

Joint Posterior with uninformative priors for x_0 and $\boldsymbol{\Omega}$

$$\underbrace{\Pi(\boldsymbol{\Theta}|\mathbf{z}_T)}_{Posterior} \sim \underbrace{\ell\left(\boldsymbol{\Theta}|\mathbf{z}_T\right)}_{likelihood} \cdot \underbrace{p(\boldsymbol{\Theta})}_{prior} = \underbrace{\prod_{t=1}^{T} f_z\left(z_t|x_t, \boldsymbol{\Omega}\right)}_{likelihood} \cdot \underbrace{\prod_{t=1}^{T} f_x\left(x_t|x_{t-1}, \boldsymbol{\Omega}\right) \cdot 1}_{prior}$$

Log Posterior

$$\pi(\boldsymbol{\Theta}|\mathbf{z}_T) = \log\left(\Pi(\boldsymbol{\Theta}|\mathbf{z}_T)\right)$$

$$\boldsymbol{\Theta} = (\boldsymbol{\Omega}, x_0, \mathbf{x}_T)$$

DEFINITION OF OBSERVABILITY (1)

Consider observability for **fixed** underlying value $(\mathbf{\Omega}^*, x_0^*)$ and T

FIRST STEP: Is joint posterior well-behaved?

Given fixed but arbitrary **observation vector** z_T → **maximum a posteriori estimate** exists?

$$\hat{\mathbf{\Theta}}(\mathbf{z}_T) = (\hat{\mathbf{\Omega}}(\mathbf{z}_T), \hat{x}_0(\mathbf{z}_T), \hat{\mathbf{x}}_T(\mathbf{z}_T)) = \mathrm{argmax}_{\mathbf{\Theta}} \Pi(\mathbf{\Theta}|\mathbf{z}_T)$$

Possible cases:

- One single maximum → obtain estimate directly

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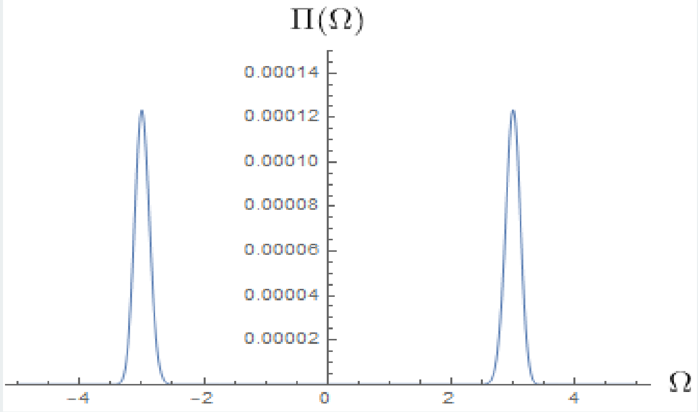
Possible cases:

- One single maximum → obtain estimate directly
- Finitely many maxima → Constrain domain of argmax

Intuitive Example

$Z_t = \Omega^2 + \epsilon_t$



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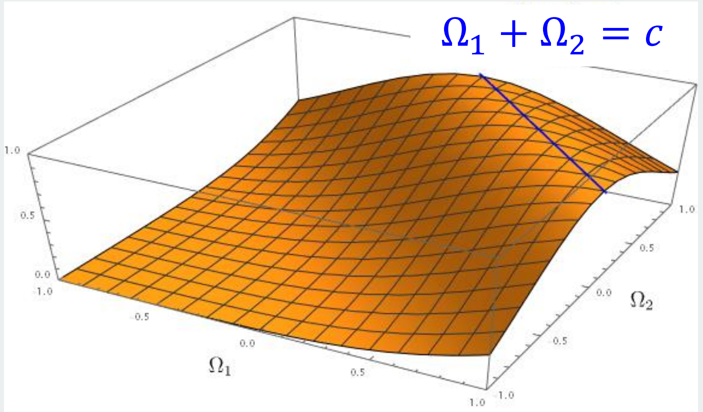
Possible cases:

- One single maximum → obtain estimate directly
- Finitely many maxima → Constrain domain of argmax
- Infinitely many maxima → No estimate can be recovered
- No maximum → No estimate can be recovered

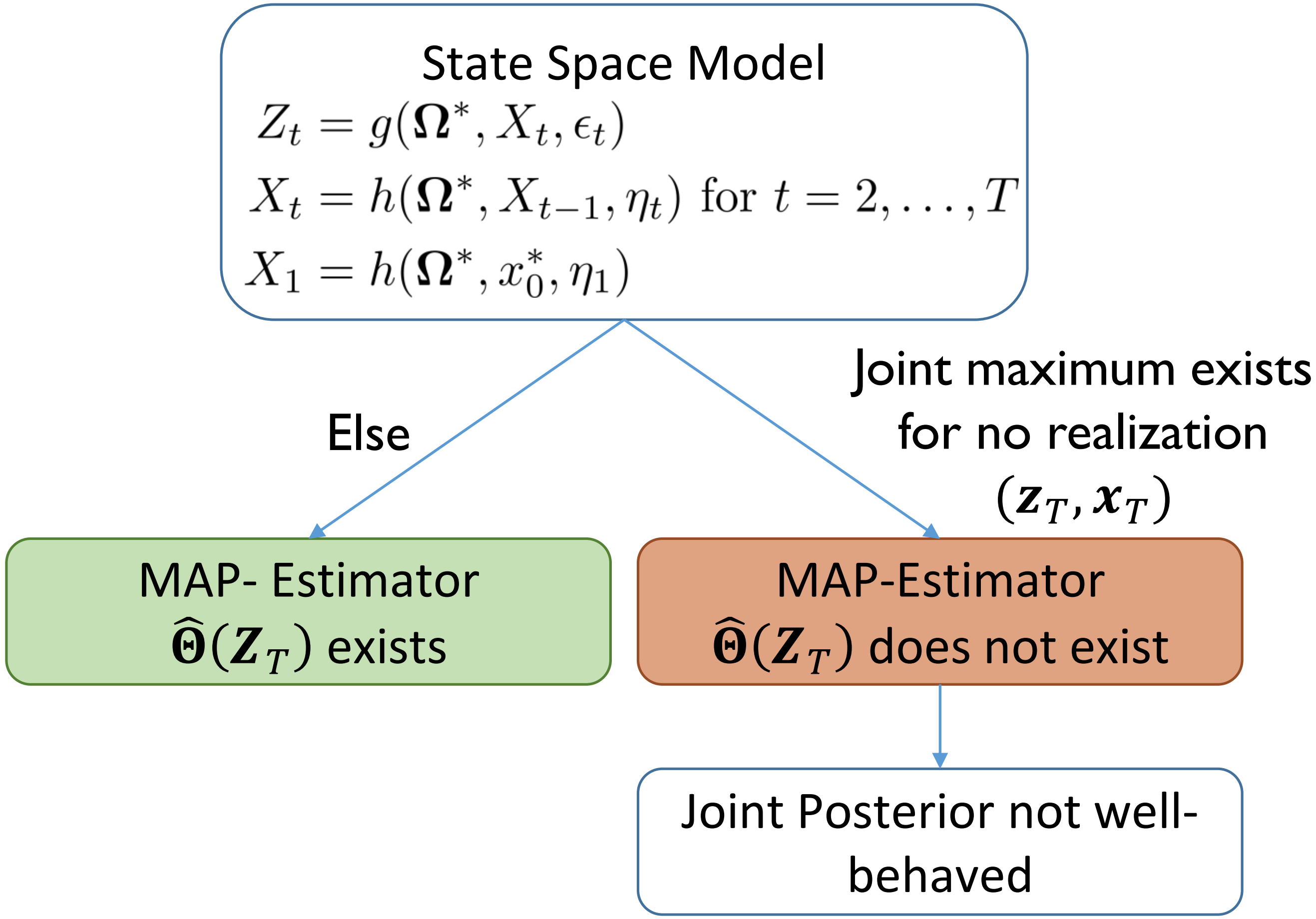
Intuitive Example

$$Z_t = \Omega_1 + \Omega_2 + \epsilon_t$$



DEFINITION OF OBSERVABILITY (2)

FIRST STEP



DEFINITION OF OBSERVABILITY (3)

Consider observability for **fixed** underlying value $(\boldsymbol{\Omega}^*, x_0^*)$ and T

SECOND STEP: Does the marginal mode exist?

- Given fixed but arbitrary **observation vector** $\boldsymbol{z}_T$
- Does maximum

$$\hat{\Omega}_j(\mathbf{z}_T)_{mar} = \operatorname{argmax}_{\Omega_j} \Pi(\Omega_j|\mathbf{z}_T) \text{ and } \hat{x}_t(\mathbf{z}_T)_{mar} = \operatorname{argmax}_{x_t} \Pi(x_t|\mathbf{z}_T)$$

$$j = 1, \ldots, |\boldsymbol{\Omega}| \qquad\qquad t = 0, \ldots, T$$

of

$$\Pi(\Omega_j|\mathbf{z}_T) = \int \Pi(\Omega, x_0, x_1, \ldots, x_T|\mathbf{z}_T)\,\mathrm{d}x_0\,\mathrm{d}x_1 \ldots \mathrm{d}x_T\,\mathrm{d}\Omega_1 \ldots \mathrm{d}\Omega_{j-1}\,\mathrm{d}\Omega_{j+1}\,\mathrm{d}\Omega_{|\Omega|}$$

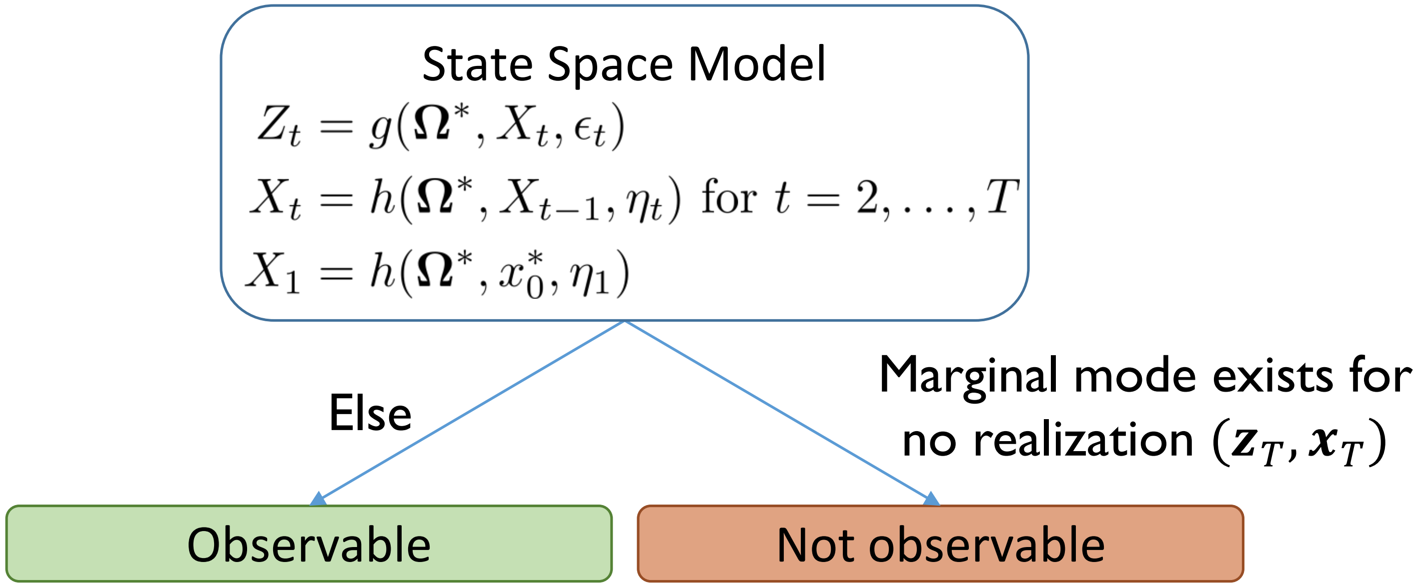
$$\Pi(x_t|\mathbf{z}_T) = \int \Pi(\Omega, x_0, x_1, \ldots, x_T|\mathbf{z}_T)\,\mathrm{d}x_0\,\mathrm{d}x_1 \ldots \mathrm{d}x_{t-1}\,\mathrm{d}x_{t+1} \ldots \mathrm{d}x_T\,\mathrm{d}\Omega_1 \ldots \mathrm{d}\Omega_{|\Omega|},$$

exist? Then, obtain marginal mode estimate

$$\hat{\boldsymbol{\Theta}}(\mathbf{z}_T)_{mar} = (\hat{\boldsymbol{\Omega}}(\mathbf{z}_T)_{mar}, \hat{x}_0(\mathbf{z}_T)_{mar}, \hat{x}_T(\mathbf{z}_T)_{mar}).$$

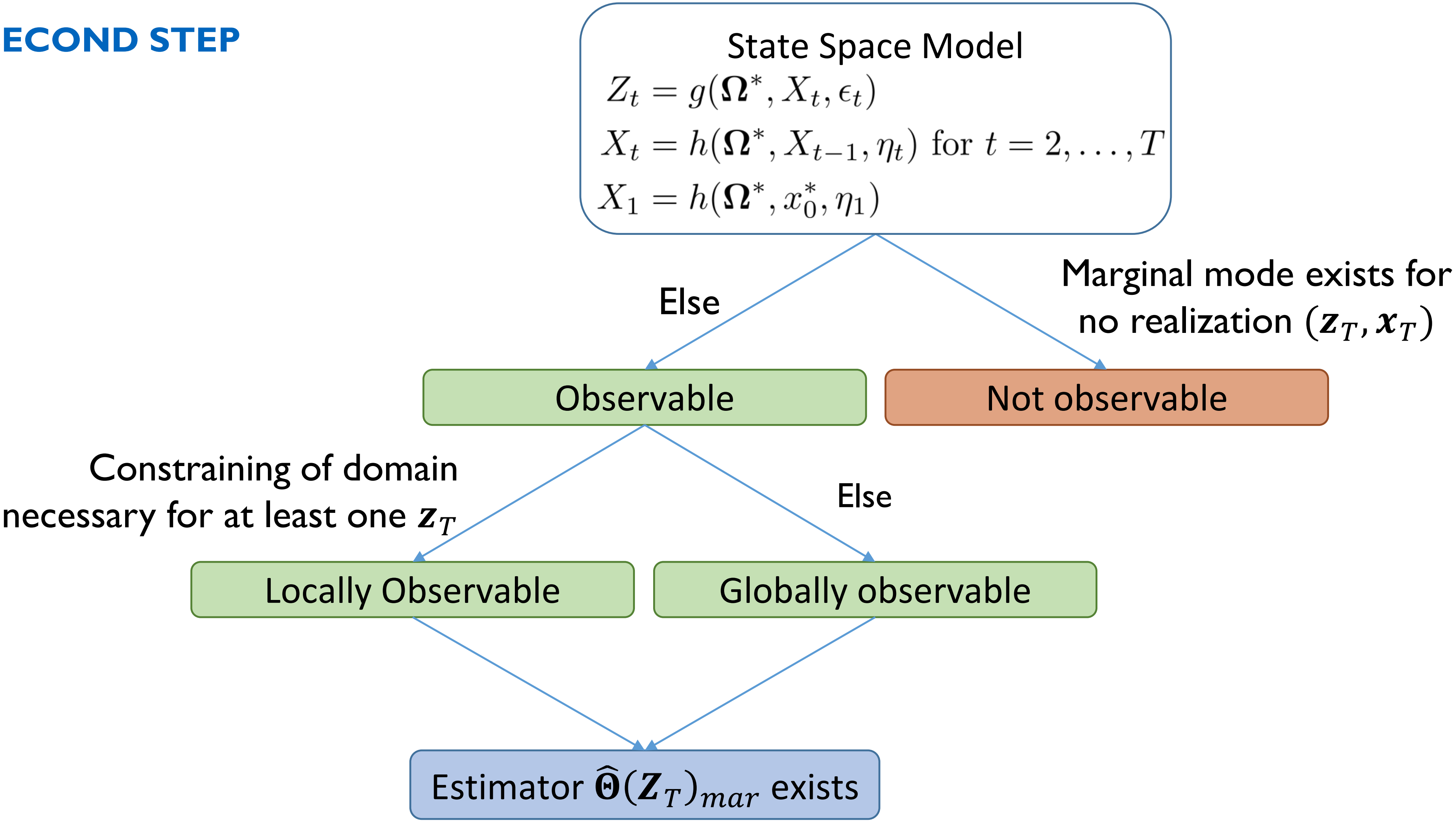
DEFINITION OF OBSERVABILITY (4)

SECOND STEP



DEFINITION OF OBSERVABILITY (4)

SECOND STEP



GOALS

- Estimators cannot be calculated analytically
- Approach to **check observability** for any kind of SSM
- As **general** as possible → to generalize it to Copula-SSM
- Not only answer question of observability but also have a **quantitative measure**

PROPOSED METHOD (I)

State-Observation Space

For $x_0^* \in \mathrm{R}$, $\mathbf{\Omega}^* \in \mathrm{R}^{|\mathbf{\Omega}|}$

$$\begin{aligned}SO(\mathbf{\Omega}^*, x_0^*) = \{&\text{Realizations of } (\mathbf{X}_T, \mathbf{Z}_T) = (X_1, \dots, X_T, Z_1, \dots, Z_T) : \\ &Z_t = g(\mathbf{\Omega}^*, X_{t-1}, \epsilon_t) \text{ for } t = 1 \dots, T\ , \\ &X_t = h(\mathbf{\Omega}^*, X_{t-1}, \eta_t) \text{ for } t = 2 \dots, T, \\ &X_1 = h(\mathbf{\Omega}^*, x_0^*, \eta_t), \\ &\epsilon_t \sim \mathcal{N}(0,1) \text{ i.i.d., and } \eta_t \sim \mathcal{N}(0,1) \text{ i.i.d. independent}\}\end{aligned}$$

PROPOSED METHOD (2)

- Fix x_0^* , $\mathbf{\Omega}^*$, and T
- Assume, $\widehat{\mathbf{\Theta}}(\boldsymbol{Z}_T)$ and $\widehat{\mathbf{\Theta}}(\boldsymbol{Z}_T)_{mar}$ exist
- Then: Estimators do **not** have to be **defined everywhere** in $SO(\mathbf{\Omega}^*, x_0^*)$
  - I.e.: For one realization $(\boldsymbol{x}_T^1, \boldsymbol{z}_T^1)$, find an estimate $\text{argmax}_{\mathbf{\Theta}} \Pi(\mathbf{\Theta}|\boldsymbol{z}_T^1)$
  - For another realization $(\boldsymbol{x}_T^2, \boldsymbol{z}_T^2)$, do not find an estimate $\text{argmax}_{\mathbf{\Theta}} \Pi(\mathbf{\Theta}|\boldsymbol{z}_T^2)$

Intuitive Example

$$Z_t = \frac{1}{\Omega} + \epsilon_t, \epsilon_t \sim N(0,1)$$

$$\widehat{\Omega}(\boldsymbol{Z}_T) = \frac{1}{\sum_{t=1}^T Z_t}$$

→ Estimator $\widehat{\mathbf{\Theta}}(\boldsymbol{Z}_T)$ exists
→ But not defined for $\boldsymbol{z}_T$ with $\sum_{t=1}^T z_t$=0

PROPOSED METHOD (2)

- Fix x_0^* , $\boldsymbol{\Omega}^*$, and T

- Assume, $\widehat{\boldsymbol{\Theta}}(\boldsymbol{Z}_T)$ and $\widehat{\boldsymbol{\Theta}}(\boldsymbol{Z}_T)_{mar}$ exist
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  - For another realization $(\boldsymbol{x}_T^2, \boldsymbol{z}_T^2)$, do not find an estimate $\text{argmax}_{\boldsymbol{\Theta}} \Pi(\boldsymbol{\Theta}|\boldsymbol{z}_T^2)$

- In **theory**: Consider all possible observation vectors $\boldsymbol{z}_T$ and corresponding $\boldsymbol{x}_T$

  → Can we find $\widehat{\boldsymbol{\Theta}}(\boldsymbol{z}_T)$ and $\widehat{\boldsymbol{\Theta}}(\boldsymbol{z}_T)_{mar}$?
- In **practice**: Not possible to consider an infinite amount of possible vectors
- Solution: **Design observation vectors** representing elements in $SO(\boldsymbol{\Omega}^*, x_0^*)$

Intuitive Example

$$Z_t = \frac{1}{\Omega} + \epsilon_t, \epsilon_t \sim N(0,1)$$

$$\widehat{\Omega}(\boldsymbol{Z}_T) = \frac{1}{\sum_{t=1}^{T} Z_t}$$

→ Estimator $\widehat{\boldsymbol{\Theta}}(\boldsymbol{Z}_T)$ exists

→ But not defined for $\boldsymbol{z}_T$ with $\sum_{t=1}^{T} z_t$=0

CONSTRUCTION OF DESIGN OBS VECTORS (1)

Consider

$$\begin{aligned} Z_t &= g(\mathbf{\Omega}^*, X_t, \epsilon_t) \\ X_t &= h(\mathbf{\Omega}^*, X_{t-1}, \eta_t) \text{ for } t = 2, \ldots, T \\ X_1 &= h(\mathbf{\Omega}^*, x_0^*, \eta_1) \end{aligned}$$

with $\epsilon_t \sim N(0,1)$ i.i.d. and $\eta_t \sim N(0,1)$ i.i.d. independent

- Want to construct design values for $\boldsymbol{Z}_T = (Z_1, \ldots, Z_T)$ and underlying $\boldsymbol{X}_T = (X_1, \ldots, X_T)$
- Part we can influence: Disturbances ϵ_t, η_t

→ Have to find suitable values for the disturbances

→ Find representative values for $N_d(\mathbf{0}, I_d)$

CONSTRUCTION OF DESIGN OBS VECTORS (2)

Goal: **approximate** density of $N_d(\mathbf{0}, I_d)$ by **discrete distribution** that takes on **M point-symmetric** values with equal probabilities

Steinbring et al. "The Smart Sampling Kalman Filter with Symmetric Samples" 11, no. 1 (2016)

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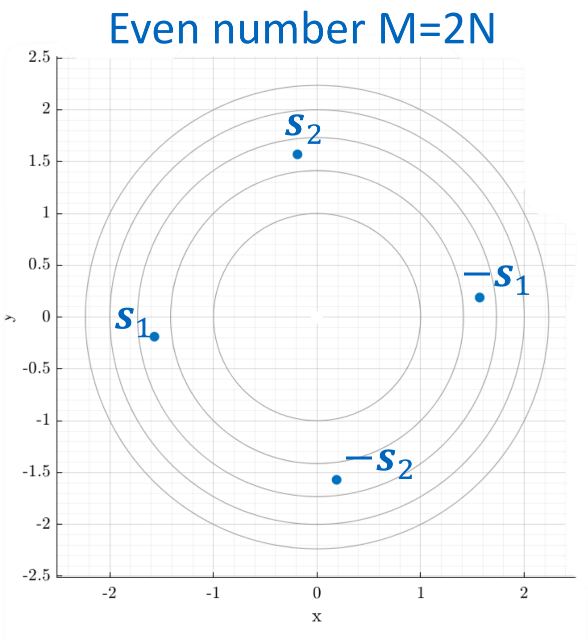

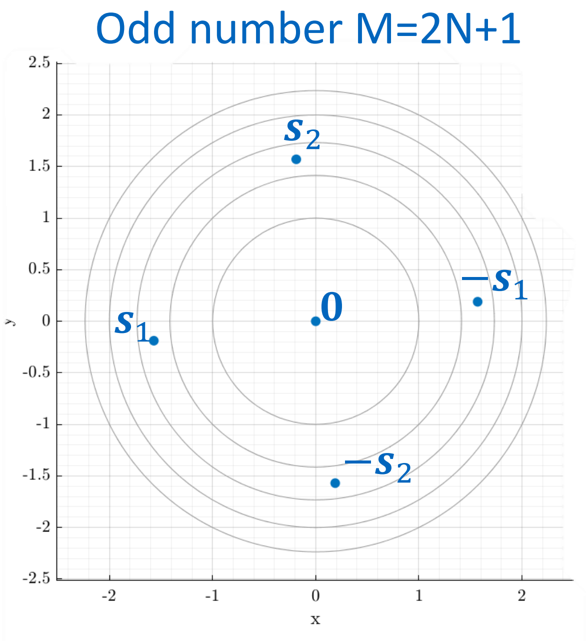

Discrete distributions with point masses $\{-\boldsymbol{s}_1, \ldots, -\boldsymbol{s}_N, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_N\}$ and $\{\mathbf{0}, -\boldsymbol{s}_1, \ldots, -\boldsymbol{s}_N, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_N\}$

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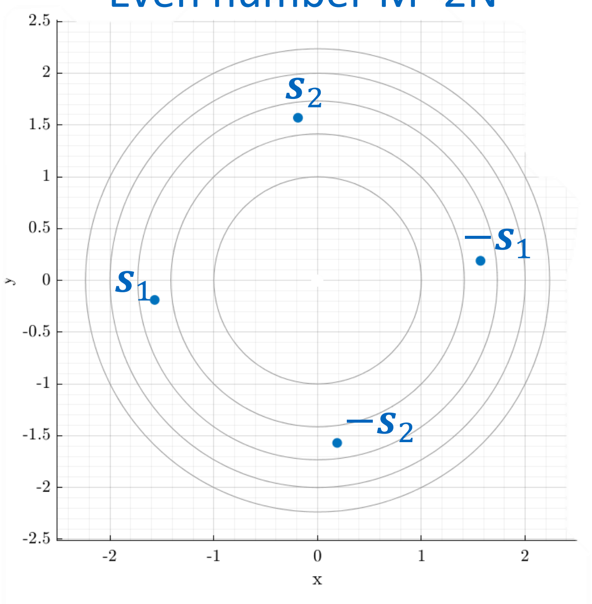
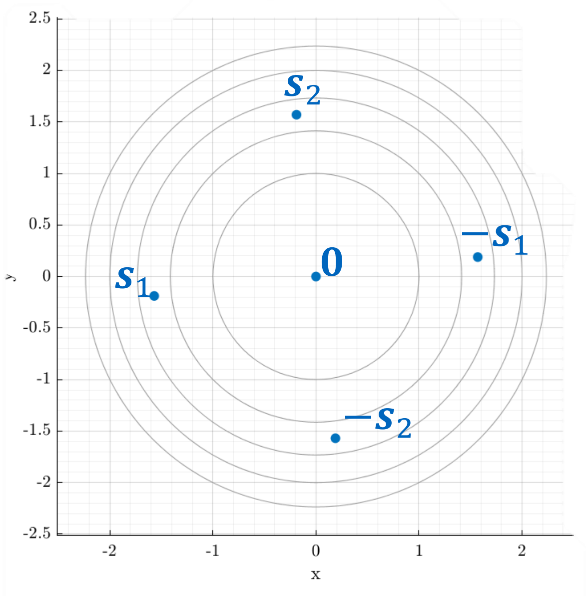
Even number M=2N



Odd number M=2N+1



Discrete distributions with point masses $\{-\boldsymbol{s}_1, \dots, -\boldsymbol{s}_N, \boldsymbol{s}_1, \dots, \boldsymbol{s}_N\}$ and $\{\mathbf{0}, -\boldsymbol{s}_1, \dots, -\boldsymbol{s}_N, \boldsymbol{s}_1, \dots, \boldsymbol{s}_N\}$

Goal: Find an optimal set $S_N^d = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_N\}$ approximating the density of $N_d(\mathbf{0}, I_d)$

→ Define distance measure $D(S_N^d)$ between $N_d(\mathbf{0}, I_d)$ and discrete distribution defined by S_N^d → $\min_{\mathcal{S}_N^d} D(\mathcal{S}_N^d)$

Steinbring et al. "The Smart Sampling Kalman Filter with Symmetric Samples" 11, no. 1 (2016)

CONSTRUCTION OF DESIGN OBS VECTORS (3)

- Want to construct K design observation vectors
- Use $(\epsilon_1, \dots, \epsilon_T, \eta_1, \dots, \eta_T)^T \sim N_{2T}(\mathbf{0}, I_{2T})$
- **Approximate** density of $N_{2T}(\mathbf{0}, I_{2T})$ by $M = K$ point masses and obtain $(\tilde{\epsilon}_1^k, \dots, \tilde{\epsilon}_T^k, \tilde{\eta}_1^k, \dots, \tilde{\eta}_T^k),\ \mathrm{k} = 1, \dots, K$
- **Insert** → Design observation vectors and corresponding design state vectors

$$\begin{aligned}
\tilde{z}_t^k &= g(\mathbf{\Omega}^*, \tilde{x}_t^k, \tilde{\epsilon}_t^k) \text{ for } t = 1, \dots, T \\
\tilde{x}_t^k &= h(\mathbf{\Omega}^*, \tilde{x}_{t-1}^k, \tilde{\eta}_t^k) \text{ for } t = 2, \dots, T \\
\tilde{x}_1^k &= h(\mathbf{\Omega}^*, x_0^*, \tilde{\eta}_1^k)
\end{aligned}$$

PROPOSED METHOD – FIRST PART

Use K design observation vectors $\left(\tilde{z}_1^k, \dots, \tilde{z}_T^k\right)_{k=1,\dots,K}$

↓

Numerically obtain $\text{argmax}_{\boldsymbol{\Theta}} \pi\left(\boldsymbol{\Theta} \middle| \tilde{\boldsymbol{z}}_T^k\right), k = 1, \dots, K$

↓

Check for all k that maximum is found → We obtain K_{max} estimates $\widehat{\boldsymbol{\Theta}}(\tilde{\boldsymbol{z}}_T^k)_{k=1,\dots,K_{max}}$

PROPOSED METHOD – FIRST PART

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Check for all k that maximum is found → We obtain K_{max} estimates $\widehat{\boldsymbol{\Theta}}(\tilde{\boldsymbol{z}}_T^k)_{k=1,\ldots,K_{max}}$

$K_{max} = 0$: $\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})$ does not exist

$K_{max} > 0$: An estimator $\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})$ corresponding to $\widehat{\boldsymbol{\Theta}}(\tilde{\boldsymbol{z}}_T^k)_{k=1,\ldots,K_{max}}$ exists

PROPOSED METHOD – SECOND PART (I)

Use K design observation vectors
$\left(\tilde{z}_1^k, \dots, \tilde{z}_T^k\right)_{k=1,\dots,K}$

↓

Numerically obtain
$\text{argmax}_{\Theta_j} \pi\left(\Theta_j \middle| \tilde{\boldsymbol{z}}_T^k\right), k = 1, \dots, K$

- Integrals cannot be calculated analytically

- Solution: **MCMC-Sampling**
- STAN: Uses extension of HMC

- Obtain marginal posterior mode estimates from **univariate kernel density estimates**

- First step needed to check if sampler will converge

PROPOSED METHOD – SECOND PART (1)

Use K design observation vectors
$\left(\tilde{z}_1^k, \dots, \tilde{z}_T^k\right)_{k=1,\dots,K}$

Numerically obtain with STAN
$\mathrm{argmax}_{\Theta_j} \pi\left(\Theta_j \middle| \tilde{\mathbf{z}}_T^k\right), k = 1, \dots, K$

Check for all k if STAN converged
→ We obtain K_{max}^{STAN} estimates
$(\widehat{\boldsymbol{\Theta}}(\tilde{\mathbf{z}}_T^k)_{mar})_{k=1,\dots,K_{max}^{STAN}}$

$K_{max}^{STAN} = 0$

Not observable
$\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})_{mar}$ does not exist

$K_{max}^{STAN} > 0$

Observable
$\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})_{mar}$ corresponding to $(\widehat{\boldsymbol{\Theta}}(\tilde{\mathbf{z}}_T^k)_{mar})_{k=1,\dots,K_{max}^{STAN}}$ exists

- Integrals cannot be calculated analytically
- Solution: **MCMC-Sampling**
- STAN: Uses extension of HMC
- Obtain marginal posterior mode estimates from **univariate kernel density estimates**
- First step needed to check if sampler will converge

PROPOSED METHOD – SECOND PART (2)

Use K design observation vectors
$\left(\tilde{z}_1^k, \dots, \tilde{z}_T^k\right)_{k=1,\dots,K}$

↓

Numerically obtain with STAN
$\text{argmax}_{\Theta_j} \pi\left(\Theta_j \middle| \tilde{\mathbf{z}}_T^k\right), k = 1, \dots, K$

↓

Check for all k if STAN converged
→ We obtain K_{max}^{STAN} estimates
$(\widehat{\boldsymbol{\Theta}}(\tilde{\mathbf{z}}_T^k)_{mar})_{k=1,\dots,K_{max}^{STAN}}$

$K_{max}^{STAN} = 0$ → **Not observable**
$\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})_{mar}$ does not exist

$K_{max}^{STAN} > 0$ → **Observable**
$\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})_{mar}$ corresponding to $(\widehat{\boldsymbol{\Theta}}(\tilde{\mathbf{z}}_T^k)_{mar})_{k=1,\dots,K_{max}^{STAN}}$ exists

Quantitative Measure

System observable → Want to classify how well

- $\text{Var}\left(\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})_{mar}\right)$
- $\text{MSE}\left(\widehat{\boldsymbol{\Theta}}(\mathbf{Z}_\mathrm{T})_{mar}\right)$
- Magnitude of K_{max}/K, K_{max}^{STAN}/K

EXAMPLE

Consider

$$Z_t = a \cdot X_t + \sqrt{1-a^2} \cdot \epsilon_t$$

$$X_t = a \cdot X_{t-1} + \sqrt{1-a^2} \cdot \eta_t, t = 1, \ldots, T$$

with $\epsilon_t \sim N(0,1)$ i.i.d. and $\eta_t \sim N(0,1)$ i.i.d. independent

Desired parameters: a and x_0; Fixed: a^*, x_0^*, T → K design vectors $\left(\tilde{\mathbf{z}}_T^k\right)_{k=1,\ldots,K} = \left(\tilde{z}_1^k, \ldots, \tilde{z}_T^k\right)_{k=1,\ldots,K}$

→ Obtain estimates

$$(\hat{a}(\tilde{\mathbf{z}}_T^k), \hat{x}_0(\tilde{\mathbf{z}}_T^k)), k = 1, \ldots, K_{max}$$

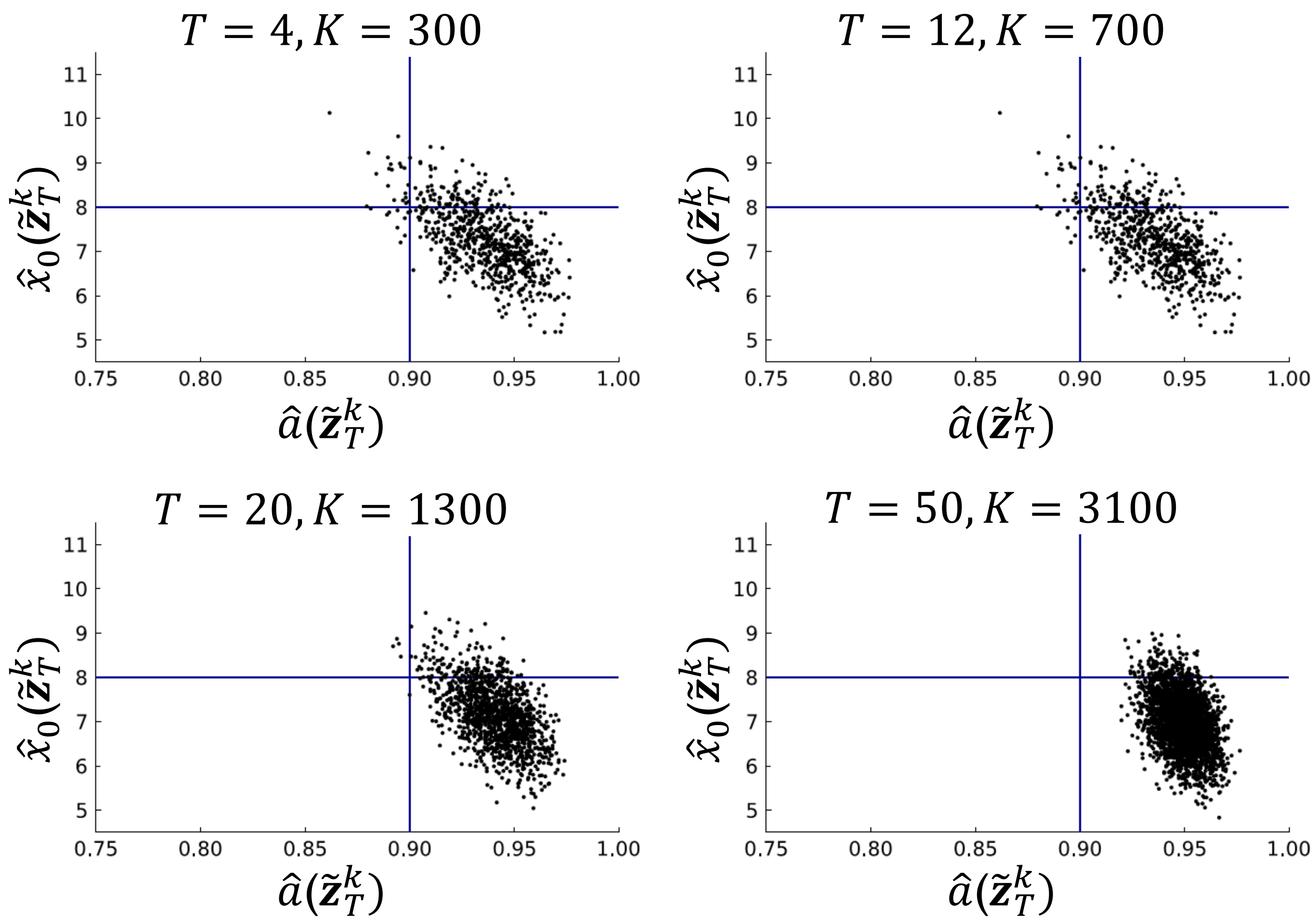
$$(\hat{a}(\tilde{\mathbf{z}}_T^k)_{mar}, \hat{x}_0(\tilde{\mathbf{z}}_T^k)_{mar}), k = 1, \ldots, K_{max}^{STAN}$$

We set, $\mathrm{T} \in \{4, 12, 20, 50\}, (\mathrm{a}^*, \mathrm{x}_0^*) = (0.9, 8)$

EXAMPLE: $(a^*, x_0^*) = (0.9,8)$

FIRST STEP

$K_{max} = K$

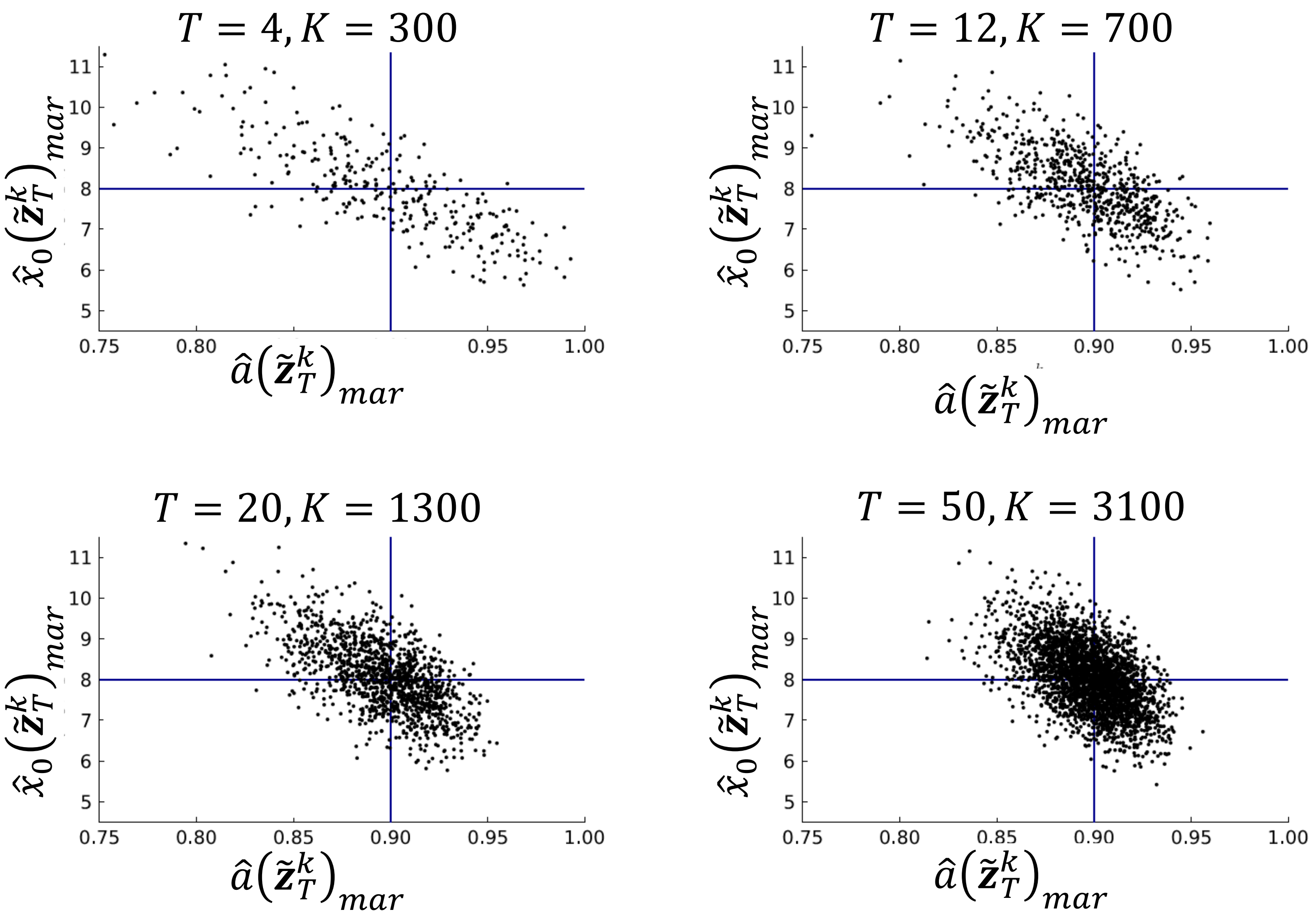


The estimates $(\hat{a}(\tilde{\mathbf{z}}_T^k), \hat{x}_0(\tilde{\mathbf{z}}_T^k)), k = 1, \dots, K_{max}$. The true value is $(a^*, x_0^*) = (0.9,8)$.

EXAMPLE: $(a^*, x_0^*) = (0.9, 8)$

SECOND STEP

$K_{max}^{STAN} = K$



The estimates $(\hat{a}(\tilde{\mathbf{z}}_T^k)_{mar}, \hat{x}_0(\tilde{\mathbf{z}}_T^k)_{mar}), k = 1, \dots, K_{max}^{STAN}$. The true value is $(a^*, x_0^*) = (0.9, 8)$.

CONCLUSIONS & OUTLOOK

Conclusions

- Developed approach to check observability of any State Space Model with Gaussian disturbances
- Easy to implement
- Do not only answer question of observability but also provide a quantitative measure

Outlook

- Adjust the deterministic samples in order to check observability of Copula-SSM
- Analyze Copula-SSM with multiple latent variables

THE DISTANCE MEASURE

Modified Cramér-von Mises

- Uses so-called localized cumulative distribution (LCD)
- Integration over finite domains
- Here: Integration over Gaussian windows

Definition LCD

- d-dimensional density function f of random vector $\boldsymbol{X}$

$$H_{\mathbf{X}}(\boldsymbol{\alpha}, \beta) = \int_{R^d} f(\mathbf{s}) K(\mathbf{s} - \boldsymbol{\alpha}, \beta) \, \mathrm{d}\mathbf{s}, \boldsymbol{\alpha} \in R^d, \beta \in R_+$$

with kernel

$$K(\mathbf{s} - \boldsymbol{\alpha}, \beta) = \exp\left[-\frac{1}{2}\frac{\|\mathbf{s} - \boldsymbol{\alpha}\|_2^2}{\beta^2}\right].$$

- $\boldsymbol{\alpha}$: Location of the kernel
- β: Size of the kernel

Conventional Cramér-von Mises

- Uses Distribution functions F and $\tilde{F}$
- Integration over half-open infinite hyperspaces

$$D_{CvM}(F, \tilde{F}) = \int_{-\infty}^{\infty} (F(\mathbf{x}) - \tilde{F}(\mathbf{x}))^2 f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

$$D(H, \tilde{H}) = \int_0^{\infty} w(b) \int_{R^d} (H(\boldsymbol{\alpha}, \beta) - \tilde{H}(\boldsymbol{\alpha}, \beta))^2 \, \mathrm{d}\boldsymbol{\alpha} \, \mathrm{d}b$$

$$w(b) = \begin{cases} \pi^{-d/2} b^{1-d} & \beta \in (0, \beta_{max}] \\ 0 & \text{else} \end{cases}$$

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