Distribution generalization in semi-parametric models: A control function approach

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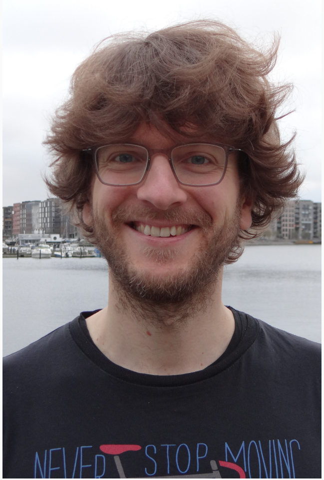

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Distribution Generalization

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- $P_{\text{train}} = P_{\text{test}}$.

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- We model $\mathcal{P}_{\text{test}}$ as

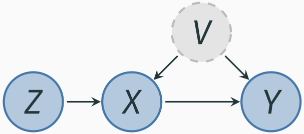
$$\mathcal{P}_{\text{test}} = \{\text{distributions generated by interventions on an SCM}\}.$$

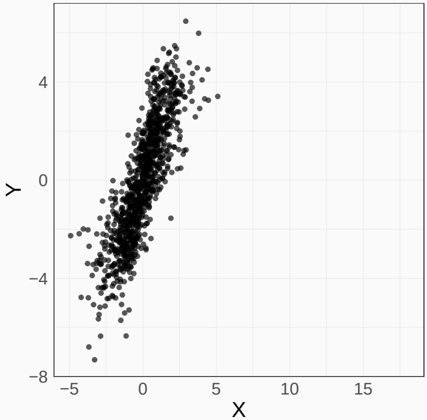
Causality and distribution generalization

$$Z \sim P_Z \perp\!\!\!\perp (V, \epsilon_Y) \sim N(0, \Sigma),$$

$$X := M_0 Z + V,$$

$$Y := f_0(X) + \gamma_0^T V + \epsilon_Y.$$





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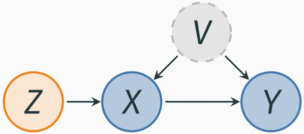

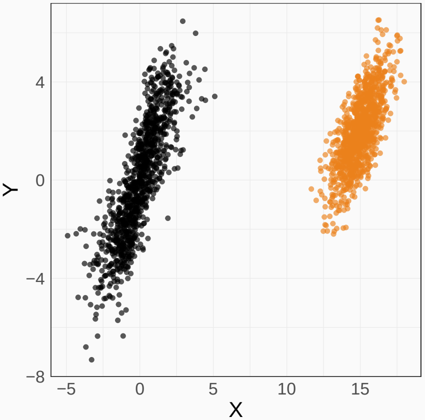

Goal

The goal of the project is to identify and learn the function that minimizes the worst-case MSE over arbitrary interventions on Z, i.e.,

$$\arg\min_{f\in\mathcal{F}} \sup_{z\in\mathbb{R}^r} \mathrm{E}\left[(Y - f(X))^2 \mid \mathrm{do}(Z := z)\right].$$

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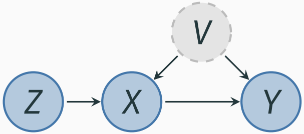

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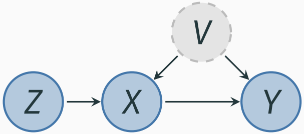

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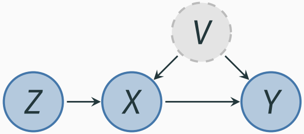

- f_0 is identified if and only if $\text{rank}(M_0) = p$, where $p =$ number of predictors.
- Control function approach [Ng and Pinkse, 1995, Newey et al., 1999].

1. $V = X - M_0 Z$
2. $E[Y | X, V] = f_0(X) + \gamma_0^T V$

When f_0 is under-identified

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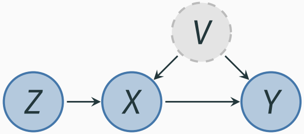

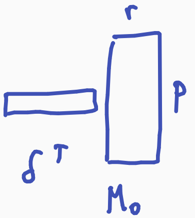

$= 0, \quad \delta \neq 0, \quad r < p$

$$\delta^T X = \underbrace{\delta^T M_0}_{0} Z + \delta^T V = \delta^T V$$

$$\Rightarrow \delta^T X - \delta^T V = 0$$

When f_0 is under-identified

Control function identifies a space of solutions.

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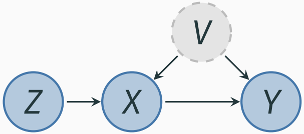

$$E[Y \mid X, V] = f_0(X) + \gamma_0^T V$$
$$= f_0(X) + \gamma_0^T V + 0$$
$$= f_0(X) + \delta^T X + \gamma_0^T V - \delta^T V$$

Space of sol'n $= \{ x \mapsto f_0(x) + \delta^T x : \delta \in \ker(M_0^T) \}$.

Pick the most predictive function

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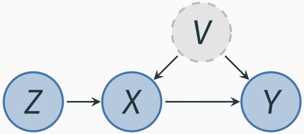

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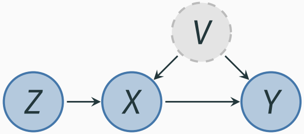

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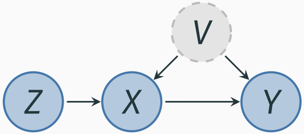

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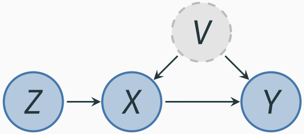

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$$\delta^* := \underset{\delta \in \ker(M_0^T)}{\arg\min}\ \mathrm{E}\left[\left(Y - f_0(X) - \delta^T X\right)^2\right].$$

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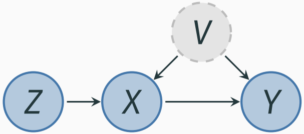

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- Resulting function is $f_0 + \delta^*$.

Properties of $f_0 + \delta^*$

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Proposition

$f_0 + \delta^*$ minimizes the worst-case MSE over arbitrary interventions on Z, i.e.,

$$f_0 + \delta^* := \arg\min_{f \in \mathcal{F}} \sup_{z \in \mathbb{R}^r} \mathrm{E}\left[(Y - f(X))^2 \mid \mathrm{do}(Z := z) \right].$$

$\mathcal{F} = \{\text{square-integrable f'n}\}$

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Remark: $f_0 + \delta^*$ can be learned with non-parametric regression methods.

↳ $E[Y \mid X, V] = \tilde{f}(X) + \tilde{\delta}^T V$

Conclusion

- In under-identified IV, we can still identify the minimax function $f_0 + \delta^*$ over arbitrary interventions on Z.

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- In under-identified IV, we can still identify the minimax function $f_0 + \delta^*$ over arbitrary interventions on Z.
- We can learn $f_0 + \delta^*$ with non-parametric regression methods.
- We are working on a modified decision tree algorithm to fit $f_0 + \delta^*$.

Thank You!

References i

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