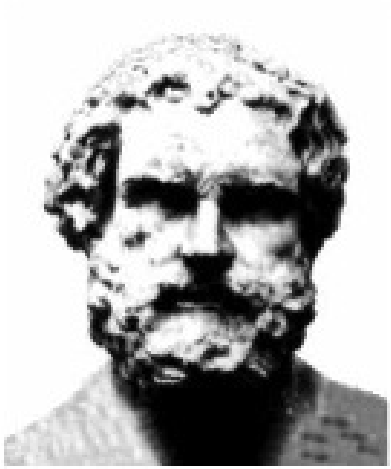
Introduction to Potential Outcomes

TUM Short Course Lecture I

Outline

- A brief History of Causation
- Potential Outcomes and Counterfactuals
- Randomized experiments
- Average Causal Effect (ACE)
- Observational Studies

Causation



Democritus (460-390 BC)

(aka the laughing philosopher because he emphasized the value of cheerfulness)

"I would rather discover a single causal relationship than be king of Persia"

The potential outcomes framework: philosophy

Hume (1748) *An Enquiry Concerning Human Understanding*:

We may define a cause to be an object followed by another, and where all the objects, similar to the first, are followed by objects similar to the second, . . .

. . . where, if the first object had not been the second never had existed.

Note: this is not one of the 3(!) causal theories Hume is famous for.

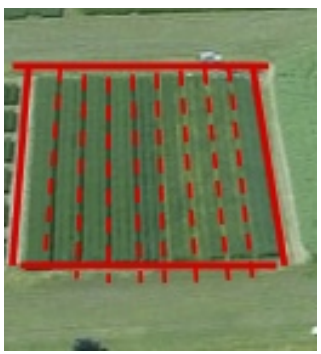
Causation

Agricultural field trials: wish to know which seed varieties produce (cause) the greatest yield... but different plots (of land) have different fertility, drainage etc.,

The potential outcomes framework: crop trials

Jerzy Neyman (1923):

To compare ν *varieties [on* $\mathfrak{m}$ *plots] we will consider numbers:*



$$\overbrace{\begin{matrix} u_{11} & \dots & u_{1m} \\ \vdots & & \vdots \\ u_{\nu 1} & \dots & u_{\nu m} \end{matrix}}^{\text{plots}} \Bigg\} \text{varieties}$$

u_{ij} is crop yield that would be observed if variety i were planted in plot j.

Physical constraints only allow one variety to be planted in a given plot in any given growing season $\Rightarrow$ Observe only one number per col.

Application to clinical trials

- Each patient in study is assigned to either:
  - Treatment (aka Drug) ($X = 1$)
  - Control (aka Placebo) ($X = 0$)
- For each patient we observe one outcome (Y), either:
  - Good e.g. Recover ($Y = 1$)
  - Bad e.g. Die ($Y = 0$)

Plots in a field $\Rightarrow$ *Patients*; *Kg of wheat* $\Rightarrow$ *Live or Die*

Potential outcomes with binary treatment and outcome

For binary treatment X, we define two potential outcome variables:

- $Y(x = 0)$: the value of Y that *would* be observed for a given unit *if* assigned $X = 0$;
- $Y(x = 1)$: the value of Y that *would* be observed for a given unit *if* assigned $X = 1$;

$Y(x = 0)$ and $Y(x = 1)$ are two different random variables (not different realizations of the same variable).

Notation: We will use $Y(x_i)$ as an abbreviation for $Y(x = i)$

Popularized by Rubin (1974); sometimes called the 'Neyman-Rubin causal model'.

Alternative notations for $Y(x = i)$ used by other authors: $Y^{x=i}$ or $Y_{x=i}$.

Potential Outcomes

Unit	Potential Outcomes	
	$Y(x=0)$	$Y(x=1)$
1	0	1
2	0	1
3	0	0
4	1	1
5	1	0

Potential Outcomes

Unit	Potential Outcomes		Observed	
	$Y(x=0)$	$Y(x=1)$	X	Y
1	0	1	1	
2	0	1	0	
3	0	0	1	
4	1	1	1	
5	1	0	0	

Potential Outcomes

Unit	Potential Outcomes		Observed	
	$Y(x=0)$	$Y(x=1)$	X	Y
1	0	1	1	1
2	0	1	0	0
3	0	0	1	0
4	1	1	1	1
5	1	0	0	1

Consistency Axiom

$$Y = (1 - X) \cdot Y(x = 0) + X \cdot Y(x = 1)$$

equivalently:

$$X = x \qquad \Rightarrow \qquad Y = Y(x).$$

In words, we have the following tautology:

For an individual who has $X = x$*, their observed response* Y *is equal to the response* $Y(x)$ *that would be observed had* X *been* x*.*

Drug Response Types:

In the simplest case where Y is a binary outcome we can think of patients as belonging to one of 4 'types':

$Y(x_0)$	$Y(x_1)$	Name
0	0	*Never Recover*
0	1	*Helped*
1	0	*Hurt*
1	1	*Always Recover*

Actual vs. Potential outcomes

Key Distinction

- X is the treatment that a given patient gets;
  thus far, this need not be randomly assigned, and could result from doctor and patient choices;

- Y is the observed response for a given patient;

- $Y(x)$ is the response that would be observed for a given paitent if (possibly counter to fact) they received $X = x$.

Potential Outcomes and Missing Data

Fundamental Problem of Causal Inference:
We never observe both $Y(x=0)$ and $Y(x=1)$.

Unit	Potential Outcomes		Observed	
	$Y(x=0)$	$Y(x=1)$	X	Y
1	?	1	1	1
2	0	?	0	0
3	?	0	1	0
4	?	1	1	1
5	1	?	0	1

Stable Unit Treatment Value Assumption (SUTVA)

- $Y(x = 0)$: the value of Y that *would* be observed for a given unit *if* assigned $X = 0$;
- $Y(x = 1)$: the value of Y that *would* be observed for a given unit *if* assigned $X = 1$;

Implicit Assumption: these outcomes, $Y(x = 0)$, $Y(x = 1)$ are 'well-defined'. Specifically:

- Only one version of $X = 1$ and $X = 0$;
  (only one version of 'drug' and 'placebo')
- Subject's outcome only depends on what they receive:
  no 'interference' between units (SUTVA).
  (Might not hold in a vaccine trial for an infectious disease if subjects are in contact.)

Average Causal Effect (ACE) of X on Y

$$\begin{aligned} \text{ACE}(X \to Y) &\equiv \mathrm{E}[Y(x_1) - Y(x_0)] \\ &= \mathrm{p}(\textit{Helped}) - \mathrm{p}(\textit{Hurt}) \qquad \in [-1, 1] \end{aligned}$$

Thus $\text{ACE}(X \to Y)$ is the difference in % recovering if everybody treated ($X = 1$) vs. if nobody treated ($X = 0$).

Identification of the ACE under randomization

If X is assigned randomly then

$$X \perp\!\!\!\perp Y(x_0) \quad \text{and} \quad X \perp\!\!\!\perp Y(x_1) \tag{1}$$

$$\begin{aligned} P(Y(x_i) = 1) &= P(Y(x_i) = 1 \mid X = i) \quad \text{(Why?)} \\ &= P(Y = 1 \mid X = i) \quad \text{(Why?)} \end{aligned}$$

Thus:

$$\begin{aligned} \text{ACE}(X \to Y) &= E[Y(x_1) - Y(x_0)] \\ &= E[Y(x_1)] - E[Y(x_0)] \\ &= E[Y(x_1) \mid X = 1] - E[Y(x_0) \mid X = 0] \\ &= E[Y \mid X = 1] - E[Y \mid X = 0]. \end{aligned}$$

Thus if (1) holds then $\text{ACE}(X \to Y)$ is identified from $P(Y \mid X)$.

Two-way Table

Under randomization, the relationship between the counterfactual distribution $P(Y(x_0), Y(x_1))$ and the observed distributions $\{P(Y \mid x_0), P(Y \mid x_1)\}$ is:

		col sums	
		$P(Y=0 \mid X=0)$	$P(Y=1 \mid X=0)$
row sums	$P(Y=0 \mid X=1)$	$P(Y(x_0)=0, Y(x_1)=0)$	$P(Y(x_0)=1, Y(x_1)=0)$
	$P(Y=1 \mid X=1)$	$P(Y(x_0)=0, Y(x_1)=1)$	$P(Y(x_0)=1, Y(x_1)=1)$

Here $P(Y=i \mid X=j) = P(Y(x_j)=i)$ due to randomization.

Equivalently we may write this in terms of types

	$P(Y=0 \mid X=0)$	$P(Y=1 \mid X=0)$
$P(Y=0 \mid X=1)$	$P(NR)$	$P(HU)$
$P(Y=1 \mid X=1)$	$P(HE)$	$P(AR)$

Identification Problem

Want: $P(Y(x_0), Y(x_1))$; Given: $P(Y \mid X=0), P(Y \mid X=1)$

Under randomization, as before: $X \perp\!\!\!\perp Y(x_i)$ implies:

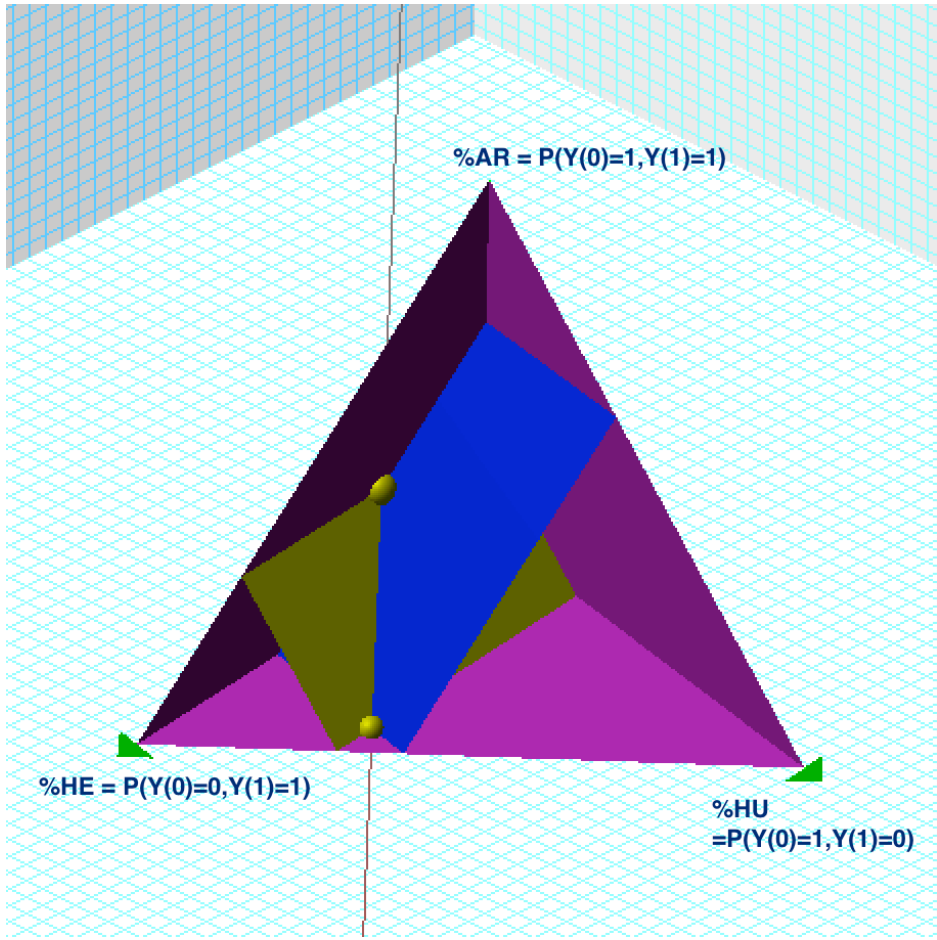
$$P(Y(x_i) = 1) = P(Y(x_i) = 1 \mid X = i) = P(Y = 1 \mid X = i).$$

Thus the observed joint $P(Y|X)$ puts two restrictions on $P(Y(x_0), Y(x_1))$:

$$\begin{aligned} P(Y=1 \mid X=0) &= P(Y(x_0)=1, Y(x_1)=0) + P(Y(x_0)=1, Y(x_1)=1) \\ P(Y=1 \mid X=1) &= P(Y(x_0)=0, Y(x_1)=1) + P(Y(x_0)=1, Y(x_1)=1). \end{aligned}$$

Each restriction implies a 2-d subset in Δ_3.
Intersection forms a 1-d subset on which ACE is constant.

Graphing Calculator Plot

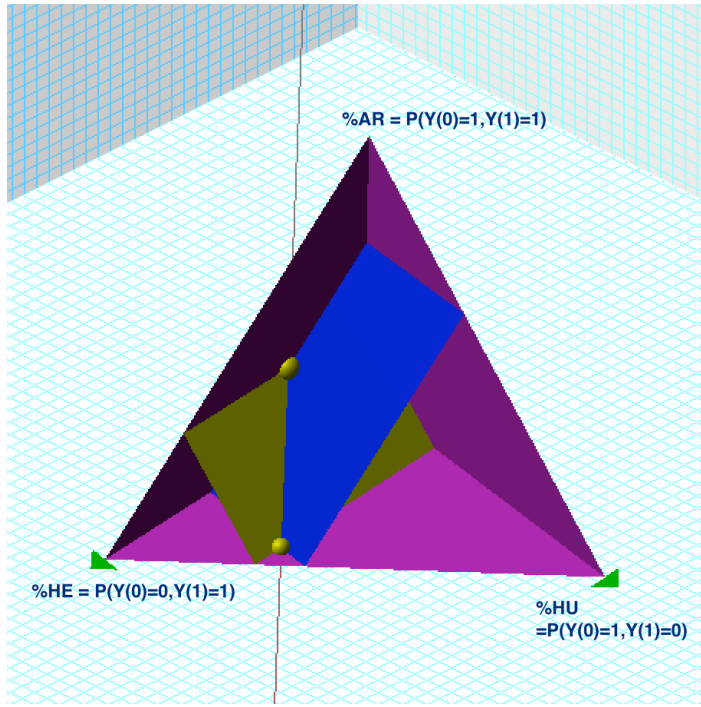


In this plot:

$P(Y=1 \mid X=0) = P(Y(x_0) = 1) = \%HU + \%AR = 0.3$, (yellow)

$P(Y=1 \mid X=1) = P(Y(x_1) = 1) = \%HE + \%AR = 0.6$, (blue)

Fréchet inequalities



Equation for line segment in simplex:

$$\left\{ \begin{array}{rcl} P(1,1) & = & t \\ P(1,0) & = & c_0 - t \\ P(0,1) & = & c_1 - t \\ P(0,0) & = & 1 - c_0 - c_1 + t \end{array} \right.$$

$$t \in \left[\max\{0, (c_0 + c_1) - 1\}, \min\{c_0, c_1\}\right]$$

$$c_0 \equiv P(Y=1 \mid X=0)$$

$$c_1 \equiv P(Y=1 \mid X=1)$$

Extreme points are given by 'Fréchet inequalities'.

Example of Fréchet inequalities

$$\left\{\begin{array}{rcl} P(1,1) &=& t \\ P(1,0) &=& c_0 - t \\ P(0,1) &=& c_1 - t \\ P(0,0) &=& 1 - c_0 - c_1 + t \end{array}\right\} \quad t \in \left[\max\{0, (c_0 + c_1) - 1\}, \min\{c_0, c_1\}\right]$$

$$c_0 \equiv P(Y=1 \mid X=0)$$
$$c_1 \equiv P(Y=1 \mid X=1)$$

Q: Suppose in a large RCT, 30% survive with Placebo, and 60% survive with Treatment, find bounds on the % Helped and Hurt

A: $c_0 = 0.3, c_1 = 0.6$

$\Rightarrow t \in \left[\max\{0, 0.3 + 0.6 - 1\}, \min\{0.3, 0.6\}\right] = [0, 0.3].$

$\%HE = P(Y(x_0)=0, Y(x_1)=1) \in [c_1 - 0.3, c_1 - 0] = [0.3, 0.6]$,
so $0.3 \leqslant \%HE \leqslant 0.6$.

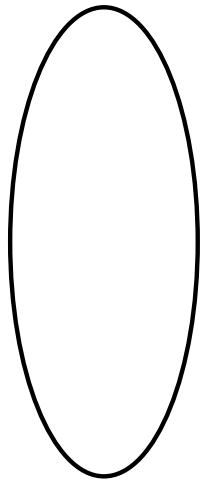
$\%HU = P(Y(x_0)=1, Y(x_1)=0) \in [c_0 - 0.3, c_0 - 0] = [0, 0.3]$

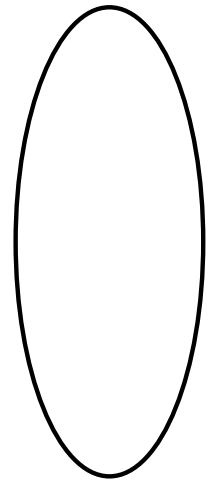
Q: Explain why $\%HE$ and $\%HU$ are not identified but $ACE = \%HE - \%HU$ is identified.

Hint: $ACE = (\%HE + \%AR) - (\%HU + \%AR)$

Big Picture: Connecting Distributions in Experiment

Counterfactual *Observed*





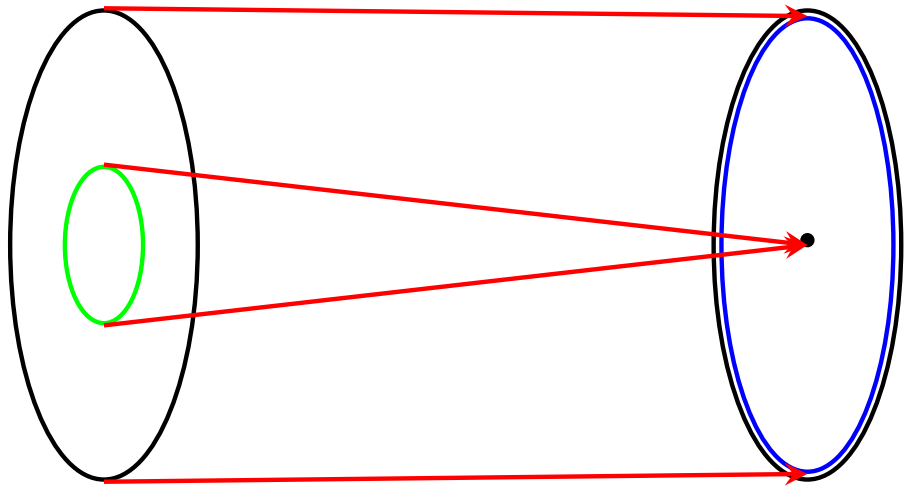
$$\Delta_3 \quad \rightarrow \quad \Delta_1 \times \Delta_1$$

$$P(Y(x_0), Y(x_1)) \quad \mapsto \quad \{P(Y \mid x_0), P(Y \mid x_1)\}$$
$$= \{P(Y(x_0)), P(Y(x_1))\}$$
(by Randomization)

$$\{\%HE, \%HU, \%NR, \%AR\} \quad \mapsto \quad \{\%HU + \%AR,\ \%HE + \%AR\}$$

Identification Problem under Experiment

Counterfactual *Observed*



$$\Delta_3 \rightarrow \Delta_1 \times \Delta_1$$

$$P(Y(x_0), Y(x_1)) \mapsto \{P(Y \mid x_0), P(Y \mid x_1)\} = \{P(Y(x_0)), P(Y(x_1))\}$$

(by Randomization)

$$\{\%HE, \%HU, \%NR, \%AR\} \mapsto \{\%HU + \%AR,\ \%HE + \%AR\}$$

Observational study; no randomization

Suppose that we do not know that $X \perp\!\!\!\perp Y(x_0)$ and $X \perp\!\!\!\perp Y(x_1)$.
What can be inferred about the ACE?

P(X,Y)	*Placebo* $X = 0$	*Drug* $X = 1$
Die: $Y = 0$	7/20	4/20
Live: $Y = 1$	3/20	6/20

What is:

- The largest proportion of people of type *Helped*, $P(Y(x_0)=0, Y(x_1)=1)$? $(6+7)/20 = 0.65$
- The smallest proportion of people of type *Hurt*, $P(Y(x_0)=1, Y(x_1)=0)$? 0

  $\Rightarrow$ Max value of ACE: $(6+7)/20 - 0 = 0.65$

Similar logic:

$\Rightarrow$ Min value of ACE: $0 - (4+3)/20 = -0.35$

(Note, as before, $P(Y = 1 \,|\, X = 0) = 0.3$, $P(Y = 1 \,|\, X = 1) = 0.6$.)

Inference for the ACE without randomization

Suppose that we do not know that $X \perp\!\!\!\perp Y(x_0)$ and $X \perp\!\!\!\perp Y(x_1)$.

What can be inferred from the observed distribution $P(X, Y)$?

General case:

$$\begin{aligned} -(P(X=0, Y=1) + P(X=1, Y=0)) \\ \leqslant \quad ACE(X \to Y) \\ \leqslant P(X=0, Y=0) + P(X=1, Y=1) \end{aligned}$$

$\Rightarrow$ Bounds will always include zero.

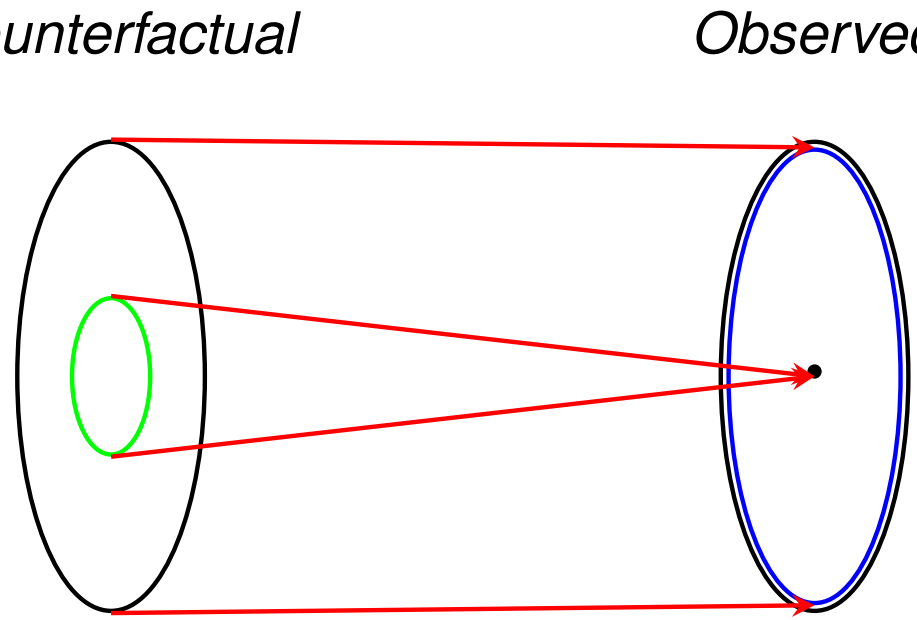
What further information can we obtain?

Observational study: one-way table!

Observed	Counterfactual	
$p(X=0, Y=0)$	$p(X=0, Y(x_0)=0, Y(x_1)=0)$	$p(X=0, Y(x_0)=0, Y(x_1)=1)$
$p(X=0, Y=1)$	$p(X=0, Y(x_0)=1, Y(x_1)=0)$	$p(X=0, Y(x_0)=1, Y(x_1)=1)$
$p(X=1, Y=0)$	$p(X=1, Y(x_0)=0, Y(x_1)=0)$	$p(X=1, Y(x_0)=1, Y(x_1)=0)$
$p(X=1, Y=1)$	$p(X=1, Y(x_0)=0, Y(x_1)=1)$	$p(X=1, Y(x_0)=1, Y(x_1)=1)$

Observed	Counterfactual	
$p(X=0, Y=0)$	$p(X=0, \text{NR})$	$p(X=0, \text{HE})$
$p(X=0, Y=1)$	$p(X=0, \text{HU})$	$p(X=0, \text{AR})$
$p(X=1, Y=0)$	$p(X=1, \text{NR})$	$p(X=1, \text{HU})$
$p(X=1, Y=1)$	$p(X=1, \text{HE})$	$p(X=1, \text{AR})$

Identification Problem



$$\Delta_7 \quad \rightarrow \quad \Delta_3$$

$$P(X, Y(x_0), Y(x_1)) \quad \mapsto \quad P(X, Y)$$

Wish to know set of $P(Y(x_0), Y(x_1))$ margins of distns $P(X, Y(x_0), Y(x_1))$ mapping to a given observed distribution $P(X, Y)$.

Want: $P(Y(x_0), Y(x_1))$; Given: $P(X, Y)$

Bounds on joints $P(Y(x_0), Y(x_1))$

Observed	Counterfactual	
$p(X=0, Y=0)$	$p(X=0, NR)$	$p(X=0, HE)$
$p(X=0, Y=1)$	$p(X=0, HU)$	$p(X=0, AR)$
$p(X=1, Y=0)$	$p(X=1, NR)$	$p(X=1, HU)$
$p(X=1, Y=1)$	$p(X=1, HE)$	$p(X=1, AR)$

$$
\begin{aligned}
0 \leqslant \quad \%HE \quad &\leqslant P(X=0, Y=0) + P(X=1, Y=1) \\
0 \leqslant \quad \%HU \quad &\leqslant P(X=0, Y=1) + P(X=1, Y=0) \\
0 \leqslant \quad \%NR \quad &\leqslant P(X=0, Y=0) + P(X=1, Y=0) \;=\; P(Y=0) \\
0 \leqslant \quad \%AR \quad &\leqslant P(X=0, Y=1) + P(X=1, Y=1) \;=\; P(Y=1)
\end{aligned}
$$

Bounds on margins $P(Y(x_i))$

Observed	Counterfactual	
$p(X=0, Y=0)$	$p(X=0, NR)$	$p(X=0, HE)$
$p(X=0, Y=1)$	$p(X=0, HU)$	$p(X=0, AR)$
$p(X=1, Y=0)$	$p(X=1, NR)$	$p(X=1, HU)$
$p(X=1, Y=1)$	$p(X=1, HE)$	$p(X=1, AR)$

We also have the following inequalities on the marginals:

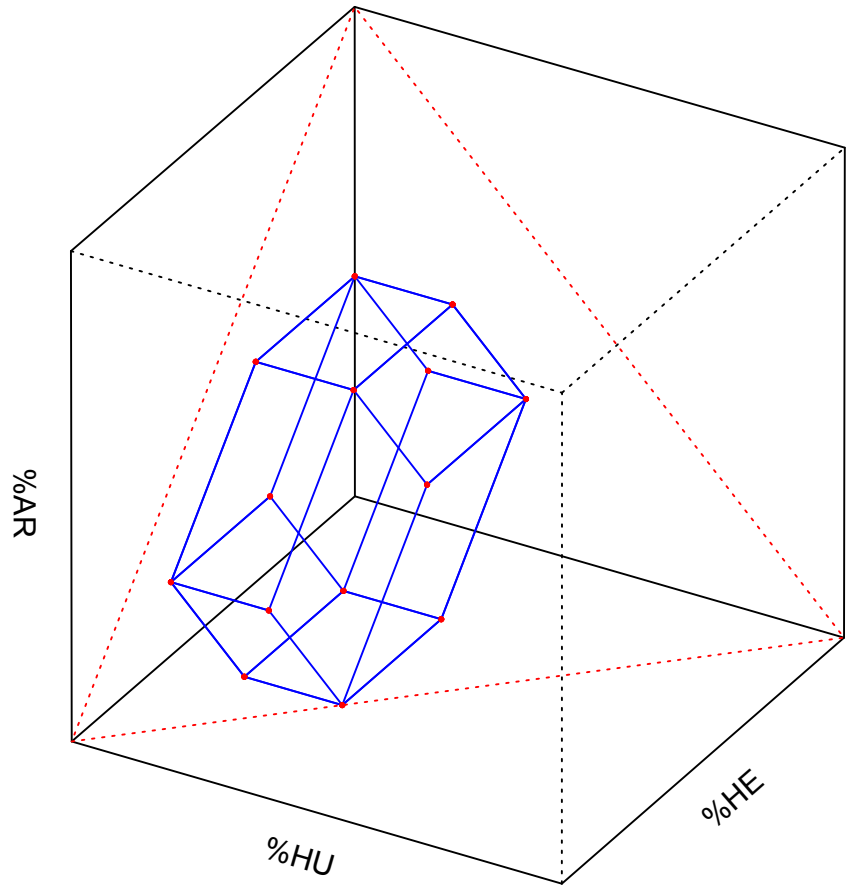
$$P(Y(x_0) = 1) = P(HU) + P(AR)$$
$$P(Y(x_1) = 1) = P(HE) + P(AR)$$

$$P(X = 0, Y = 1) \leqslant P(Y(x_0) = 1) \leqslant 1 - P(X = 0, Y = 0)$$
$$P(X = 1, Y = 1) \leqslant P(Y(x_1) = 1) \leqslant 1 - P(X = 1, Y = 0)$$

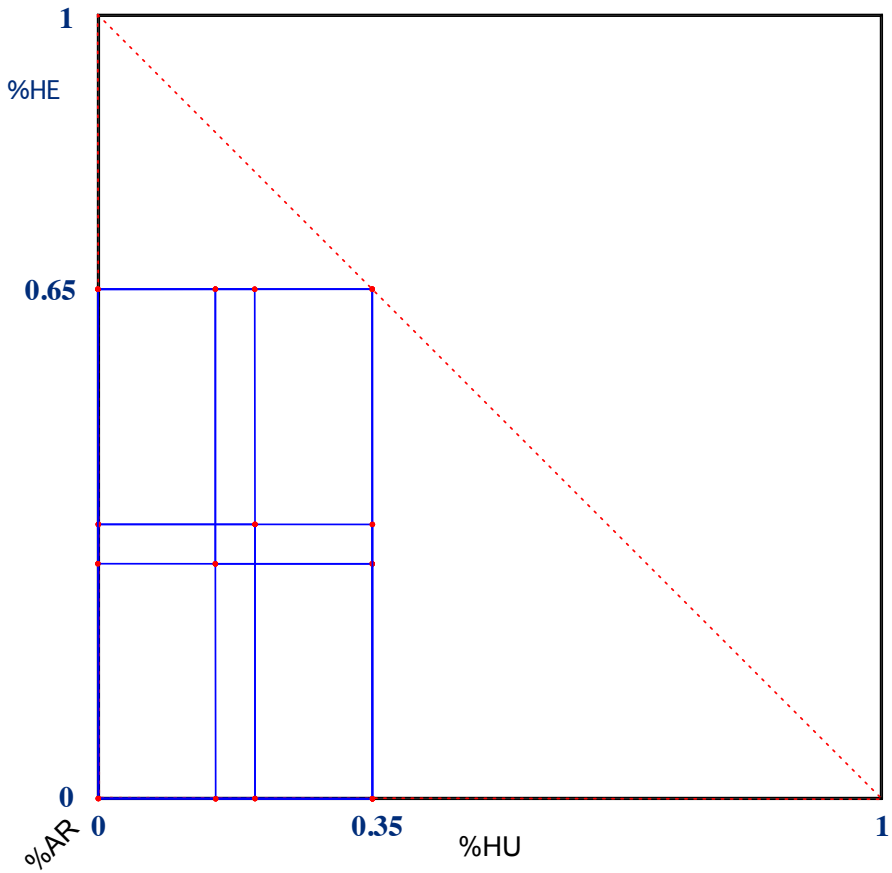
Thus we have 6 pairs of parallel planes.

Polytope for observational study

Set of margins $P(Y(x_0), Y(x_1))$ compatible with the Obs. Study.



Checking ACE bounds



This confirms the ACE bounds we derived earlier.

(But why is this helpful!?)

Summary so far

- Causal contrasts compare the *potential* outcomes of the same units under different treatments.
- In our observed data, for each unit one outcome will be 'actual'; the others will be 'counterfactual'.
  (Exceptions in fields where cross-over designs are possible.)
- The potential outcome framework allows
  Causation to be 'reduced' to *Missing Data*
  $\Rightarrow$ Conceptual progress!
- The ACE is identified if $X \perp\!\!\!\perp Y(x_i)$ for all values x_i.
- Randomization of treatment assignment implies $X \perp\!\!\!\perp Y(x_i)$.
- Without independence the ACE is not identified, and cannot be bounded away from zero.