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Ogata's construction of an index for SPT phases of quantum spin chains

Bruno Nachtergaele (UC Davis)

Outline

- Quasi-adiabatic evolution
- Stability spectral gaps
- Gapped ground state phases
- Ogata's construction of an index for SPT phases

The quasi-adiabatic evolution a.k.a. the spectral flow

Consider (Γ, d) and $\mathcal{H}_x = \mathbb{C}^{n_x}$, as before.

We will use a class of F-functions of the form $F(r) = F_0(r)e^{-ar^\theta}$, $a > 0, \theta \in (0, 1]$. Let $\mathcal{B}_{a,\theta}$ be the Banach space of interactions with finite F-norm.

Suppose Φ_0 and Φ_1 are two interactions with an interpolating continuously differentiable curve $\Phi(s)$, $s \in [0, 1]$. We assume $s \mapsto \Phi(s, X)$ is C^1 and

$$\sup_{x,y\in\Gamma} \frac{1}{F(d(x, y))} \sum_{\substack{X\in\mathcal{P}_0(\Gamma)\\ x,y\in X}} \|\Phi(X, s)\| + |X|\|\Phi'(X, s)\| < \infty.$$

Let $\mathcal{B}^1_{a,\theta}([0, 1])$ denote the Banach space of such curves of interactions.

Locality, Quasi-Locality, Almost-Locality

Locality is a crucial notion for many-body systems. Observables in $\mathcal{A}_{\mathrm{loc}}$ are called local, those in $\mathcal{A}_\Gamma$ quasi-local.

By construction, for all $A \in \mathcal{A}_\Gamma$ and any sequence $\Lambda_n \uparrow \Gamma$, there exist $\mathcal{A}_{\Lambda_n} \ni A_n \to A$. A concrete sequence of local approximations of any $A \in \mathcal{A}_\Gamma$ can be obtained by using the conditional expectations Π_Λ:

$$\Pi_\Lambda = \mathrm{id}_{\mathcal{A}_\Lambda} \otimes \rho \restriction_{\mathcal{A}_{\Gamma \setminus \Lambda}}, \text{ where } \rho \text{ is the tracial state.}$$

For any f, positive and decreasing to 0, we can define

$$\|A\|_f = \|A\| + \sup_n f(n)^{-1} \|A - \Pi_{\Lambda_n}(A)\|.$$

Then, $\mathcal{A}_f = \{A \in \mathcal{A}_\Gamma | \|A\|_f < \infty\}$ is a Banach *-algebra for this norm (e.g., Moon-Ogata 2019).

Lieb-Robinson bounds provide an estimate for commutators. Due to the following inequalities, they are useful to measure locality of observables:

$$\|A - \Pi_\Lambda(A)\| \leq \sup_{B \in \mathcal{A}_{\Gamma \setminus \Lambda}, \|B\|=1} \|[A, B]\| \leq 2\|A - \Pi_\Lambda(A)\|.$$

Telescopic sums: given $\Lambda_n \uparrow \Gamma$ and decay function f, $A \in \mathcal{A}_f$, consider the identity

$$A = \Pi_0(A) + \left[\sum_{n=1}^{N} \Pi_n(A) - \Pi_{n-1}(A)\right] + A - \Pi_N(A)$$

Note $\|\Pi_n(A) - \Pi_{n-1}(A)\| \leq \|A\|_f(f(n) + f(n-1))$. If f is summable, we obtain an absolutely convergent series for A:

$$A = \Pi_0(A) + \sum_{n=1}^{\infty}[\Pi_n(A) - \Pi_{n-1}(A)].$$

The Hastings generator of the 'quasi-adiabatic evolution' (Hastings 2004, Hastings-Wen 2005, Bachmann-Michalakis-N-Sims 2012) is defined by the 'interaction'

$$\tilde{\Psi}_x(s) = \int_{-\infty}^{\infty} w_a(t) \int_0^t \tau_u^{\Phi(s)} \left(\frac{d}{ds} \Phi_x(s) \right) du\, dt$$

with $w_a(t)$ a specific function of almost exponential decay $\sim e^{-\frac{a|t|}{(\log a|t|)^2}}$, with $a > 0$.

Using LRBs, we can show $\tilde{\Psi}_x(s) \in \mathcal{A}_f$, for a stretched exponential f.

Using a telescopic sum and conditional expectations $\Pi_{b_x(n)}$, we can construct a true interaction $\Psi \in \mathcal{B}_{a',\theta}([0,1])$, equivalent to $\tilde{\Psi}$.

Theorem (Bachmann, Michalakis, N, Sims, 2012)

(i) The automorphisms α_s *for* $s \in [0, 1]$*, generated by* $\Psi(s)$ *with* s *as the 'time'-parameter, are a strongly continuous cocycle of quasi-local automorphisms, satisfying Lieb-Robinson bounds with* F *of stretched exponential decay.*
(ii) If, in addition, Φ_0 *and* Φ_1 *and the interpolating differentiable curve* $\Phi(s)$ *are interactions with a unique gapped ground state* ω_s *with gap* $\geq \gamma > 0$*, and we pick* $a < 2\gamma/7$ *in* w_a*, we have* $\omega_s = \omega_0 \circ \alpha_s, \quad s \in [0, 1]$.

- Lieb-Robinson bounds are essential to construct true interaction and to show existence of the thermodynamic limit.
- α_s inherits any symmetries of the curve $\Phi(s)$.
- Uniqueness of the ground state can be relaxed.
- Decay classes other than stretched exponentials have been considered.

Stability of Spectral Gaps

Stability of the bulk gap

Suppose $\{h_x\}_{x\in\Gamma}$ defines generator δ with (for simplicity) a unique ground state ω and a gap $\gamma_0 > 0$:

$$\omega(A^*\delta(A)) \geq \gamma_0\omega(A^*A), A \in \operatorname{dom}\delta, \text{ with } \omega(A) = 0 \Leftrightarrow \operatorname{gap}(H_\omega) \geq \gamma_0.$$

Define perturbations of the form

$$h_x(s) = h_x + s\Phi_x, s \in \mathbb{R}, \Phi_x = \sum_n \Phi(b_x(n)), \text{ with } \|\Phi(b_x(n))\| \leq g(n).$$

The gap of the model is stable under such perturbations if for all $\gamma \in (0, \gamma_0)$, there exists $s_0(\gamma) > 0$ such that the gap for the perturbed model, γ_s, satisfies

$$\gamma_s \geq \gamma, \text{ for all } |s| < s_0(\gamma).$$

Stability theorem for frustration free finite range interactions

We consider perturbations of finite-range (R) frustration-free models with Hamiltonians of the form

$$H_\Lambda(s) = \sum_{x\in\Lambda} h_x + s \sum_{x\in\Lambda, n\geq 0} \Phi(b_x(n)) = \sum_{X\subset\Lambda} \Phi(s, X).$$

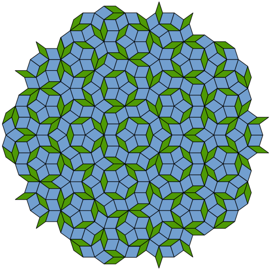
with uniformly bounded $h_x \in \mathcal{A}_{b_x(R)}$, $\sup_x \|h_x\| < \infty$. $\Gamma \subset \mathbb{R}^\nu$, Delone.
C1: There are $C > 0, q \geq 0$ such that $\mathrm{gap}(H_{b_x(n)}(0)) \geq Cn^{-q}$ (non-zero edge modes do not vanish faster than a power law).
C2: $\mathrm{gap}(H_{\omega_0}) = \gamma_0 > 0$.
C3: $\|\Phi(b_x(n))\| \leq \|\Phi\| e^{-an^\theta}$, for some $a > 0, \theta > 0$.
C4: LTQO. Denote by P_Λ the projection onto $\ker H_\Lambda(0)$. There exists a positive decreasing function G_0 for which, for all $A \in \mathcal{A}_{b_x(k)}$,

$$\|P_{b_x(m)} A P_{b_x(m)} - \omega_0(A) P_{b_x(m)}\| \leq \|A\|(k+1)^\nu G_0(m-k).$$

and

$$\sum_{n\geq 1} n^p G_0(n) < \infty, \text{ some } p > 4\nu + q$$

Not assuming a uniform gap in finite volume!



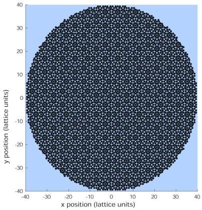


FIG. 2. Round section of the quasilattice, of radius 40 lattice units.

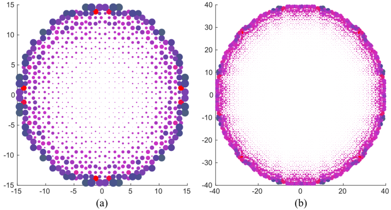


FIG. 7. (a) The radius is 15, with eigenvalue 0.018. (b) The radius is 40 with eigenvalue 0.007.

Figure: Penrose tiling. Ammann-Beenker tiling. Edges state or not? (T. Loring, J. Math. Phys. **60**, 081903 (2019))

Theorem (Stability of the bulk gap, N-Sims-Young, arXiv:2102.07209)

If conditions C1-C4 are satisfied, then, for all $\gamma \in (0, \gamma_0)$*, there is a constant* $\beta > 0$*, such that the ground state,* ω_s*, for* $\Phi(s)$*, with*

$$|s| \leq \frac{\gamma_0 - \gamma}{\beta \gamma_0}$$

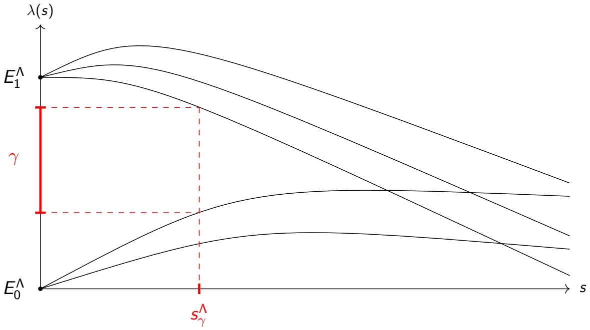
is unique, and the gap of $H_{\omega_s} > \gamma$*. For* β *can take*

$$\beta = C_0 \sum_n n^{q+\nu} G_0(n).$$

Proved using the strategy of Bravyi-Hastings-Michalakis (2010) and Michalakis-Zwolak (2013), applied to the GNS Hamiltonian.

For a model with a gap above the ground state to represent a gapped phase, the gap should be stable under a broad class of perturbations.

$$H_\Lambda(s) = H_\Lambda(0) + sV_\Lambda$$



The spectral gap of $H_\Lambda(s)$ above the 'ground state' is at least γ for all $0 \leq s \leq s_\gamma^\Lambda$.

Stability means that there is a Λ-independent lower bound for s_γ^Λ.

Gapped Ground State Phases

Consider, for a fixed choice of Γ and $n_x, x \in \Gamma$, the set $\mathcal{B}_{a,\theta}^{\mathrm{gapped}}$ of all interactions $\Phi \in \mathcal{B}_{a,\theta}$, some $a > 0, \theta \in (0,1]$, such that δ^{Φ} has a finite set of gapped ground states. Further restrictions can be imposed (uniqueness, symmetries ...).

Then, a gapped ground state phase is an equivalence class for an equivalence relation defined on $\mathcal{B}_{a,\theta}^{\mathrm{gapped}}$.

Mathematical definitions

Suppose Φ_0 and Φ_1 are two interactions in the class $\mathcal{B}_{a,\theta}$, with ground state sets $\mathcal{S}^{\Phi_0}$ and $\mathcal{S}^{\Phi_1}$, respectively.

Definition 1. (Equivalence of interactions)

The interactions Φ_0 and Φ_1 belong to the same phase if there exists a differentiable curve of interactions $[0,1] \ni s \mapsto \Phi(s)$ in $\mathcal{B}_{a,\theta}$ such that

1. $\Phi(0) = \Phi_0, \Phi(1) = \Phi_1$;
2. There exists a constant $\gamma' > 0$, such that for all $s \in [0,1]$ $\Phi(s)$ has gapped ground states with gap $\gamma' > 0$.
3. There exist $a' > 0, \theta' \in (0,1]$, such that $\Phi(\cdot) \in \mathcal{B}^1_{a',\theta'}([0,1])$, defined as the Banach space of interactions for which, with $F(r) = e^{-a' r^{\theta'}} F_0(r)$,

$$\sup_{x,y \in \Gamma} \frac{1}{F(d(x,y))} \sum_{\text{finite} X : x,y \in X} \|\Phi(X,s)\| + |X| \|\Phi'(X,s)\|$$

is a bounded by a bounded measurable function of s.

Suppose $\mathcal{S}_0$ and $\mathcal{S}_1$ are two finite sets of pure states of $\mathcal{A}_\Gamma$.

Definition 2. (Equivalence of states)
The sets of states $\mathcal{S}_0$ and $\mathcal{S}_1$ are automorphically equivalent (in the stretched exponential locality class) if there exists a continuous curve of interactions $[0,1] \ni s \mapsto \Psi(s)$ such that

1. There exist $a' > 0, \theta' \in (0,1]$, such that for all $s \in [0,1]$, $\Psi(s) \in \mathcal{B}_{a',\theta'}$;
2. $[0,1] \ni s \mapsto \Psi(s)$ is piecewise continuous in the norm of $\mathcal{B}_{a',\theta'}([0,1])$;
3. The family of automorphisms $\alpha_{s,0}$ generated by $\Psi(s)$ satisfies

$$\mathcal{S}_1 = \{\omega \circ \alpha_{1,0} \mid \omega \in \mathcal{S}_0\}.$$

It is easy to show that these two definitions define equivalence relations.

We would like to define a gapped ground state phase as an equivalence class. Which definition should we use?

Theorem (Equivalence of the Defs 1 and 2, N arXiv:2205.10460)

(i) (Def 2 $\implies$ Def 1) Let $\mathcal{S}_0$ be a set of mutually disjoint pure ground states gap bounded below by $\gamma > 0$ for the dynamics with generator δ_0 defined by an interaction $\Phi_0 \in \mathcal{B}_{a,\theta}$, for some $a > 0, \theta \in (0, 1]$. If a set of states $\mathcal{S}_1$ is automorphically equivalent to $\mathcal{S}_0$ in the stretched exponential locality class, then there exists a differentiable curve of interactions of class $\mathcal{B}^1_{a',\theta'}([0, 1])$, for some $a' > 0, \theta' \in (0, 1]$, $\Phi(s), s \in [0, 1]$, with $\Phi(0) = \Phi_0$, and such that $\mathcal{S}_1$ are gapped ground states with gap bounded below by γ for the dynamics generated by $\Phi(1)$.

(ii) (Def 1 $\implies$ Def 2) Suppose $s \mapsto \Phi(s)$ is a differentiable curve of interactions of class $\mathcal{B}^1_{a,\theta}([0, 1])$, such that there exists $\gamma > 0$ and sets of mutually disjoint pure gapped ground states $\mathcal{S}_s$, $s \in [0, 1]$, with gap bounded below by γ. Then, there exists as strongly continuous curve of automorphisms α_s of class $\mathcal{B}_{a',\theta'}([0, 1])$, such that

$$\mathcal{S}_s = \{\omega \circ \alpha_{s,0} \mid \omega \in \mathcal{S}_0\}.$$

This a mathematical version of the definition of ‘gapped phase’ given in Xie Chen, Zheng-Cheng Gu, Xiao-Gang Wen, Phys. Rev. B 82, 155138 (2010).

Ogata's construction of an invariant for Symmetry Protected Topologic (SPT) Phases

SPT phases are gapped ground state phases defined by restricting to $\Phi \in \mathcal{B}_{a,\theta}^{\mathrm{gapped}}$ that share a symmetry given by a representation of a group G, and also requiring that the interpolating curves have that symmetry at every point.

Furthermore, one focusses on the trivial phase in $\mathcal{B}_{a,\theta}^{\mathrm{gapped}}$ (without symmetry condition).

The same invariant (index) was proposed by several authors:
Chen-Gu-Wen 2010-11, Pollmann-Turner-Berg-Oshikawa 2010-11, ...

Example

The AKLT chain (Affleck-Kennedy-Lieb-Tasaki 1987-88) is the spin-1 chain with nearest neighbor interaction given by

$$P^{AKLT}_{x,x+1} = \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

which is a 5-dim projection. In (Bachmann-N 2014) we constructed a C^1-curve of projections $P(s)$ such that $P(1) = P^{AKLT}$ and the model with nn interaction $P(0)$ has a unique product ground state in the TL and we show a uniform positive lower bound for the gap for $s \in [0,1]$. This implies that the AKLT chain belongs to the same phase as the model with a unique product ground state (the trivial phase).

In contrast, if we one restricts interpolations that respect spin rotation symmetry about 1 axis and an additional $\mathbb{Z}_2$ symmetry, an index argument shows that any curve connecting the AKLT model with a model in the trivial phase, must pass through a phase transition where the gap closes (Tasaki 2018, Ogata 2019-20). This implies that the AKLT chain belongs to a SPT phase distinct from the trivial phase.

The 'Chen-Gu-Wen-Pollmann-Turner-Berg-Oshikawa-Ogata' index

1. The AKLT chain

2. Ogata's general construction

Ogata's construction of an SPT invariant for quantum spin chains

Setting: $\Gamma = \mathbb{Z}$, an interaction Φ with a unique gapped ground state in the trivial phase.

Concretely, $\Phi \in \mathcal{B}_{a,\theta}^{\text{gapped}}$, and Φ is connected by a differentiable gapped path to Φ_0, defined by

$$\Phi_0(X) = 0, \text{ unless } X = \{x\}, x \in \Gamma, \text{ and } \Phi(\{x\}) = (\mathbb{1} - |0\rangle\langle 0|)_x.$$

Φ_0 has a unique gapped ground state given by the product state

$$\bigotimes_x \langle 0| \cdot |0\rangle.$$

Φ is assumed to have a local symmetry given by unitary representations $U_x(g)$ of a group G. For the infinite chain this symmetry is described by the automorphisms

$$\beta_g(A) = \left(\bigotimes_x U_x(g)^* \right) A \left(\bigotimes_x U_x(g) \right), A \in \mathcal{A}_{\mathbb{Z}}.$$

For Symmetry Protected Phases, we define equivalence by only using differentiable paths of interactions that all have the same G-symmetry.

We want an invariant for the resulting equivalence classes of G-symmetric interactions with a unique gapped ground state which is equivalent to Φ_0 (without the symmetry).

Theorem (Ogata, TAMS 2021)

There exists an $H^2(G, U(1))$-valued invariant for the SPT equivalence classes.

Coincides with the invariants given by Pollmann-Turner-Berg-Oshikawa, 2010-11 and Chen-Gu-Wen, 2010-11 in a more restricted context.

The $H^2(G, U(1))$-valued invariant classifies the projective representations of G. The proof of the theorem is by constructing such a representation.

Inspired by what we found for the AKLT chain, we look for a unitary implementation of the symmetry G on a half-chain.

Starting point: $\Phi \sim \Phi_0$ implies the existence of a gapped interpolating path $\Phi(s), s \in [0,1]$, $\Phi(0) = \Phi_0, \Phi(1) = \Phi$, and an associated interaction $\Psi(s)$, that generates the quasi-adiabatic evolution α_s. In particular

$$\omega_1 = \omega_0 \circ \alpha_1.$$

Important property: $\Psi \in \mathcal{B}_{a',\theta'}([0,1])$, i.e., of fast decay. This means $\Psi(s)$ can be decoupled by a bounded perturbation.

Define $\Gamma_L = (-\infty, 0]$ and $\Gamma_R = [1, \infty)$ and $\tilde{\Psi}(s)$ s. t.
$\Psi(s) = \tilde{\Psi}(s) + V(s)$, with

$$\tilde{\Psi}(s,X) = \Psi(s,X) \text{ if } X \subset \Gamma_L \text{ or } X \subset \Gamma_R,\ 0 \text{ otherwise.}$$

Considering $V(s)$ as a perturbation and using interaction picture gives unitaries $U(s)$ s.t.

$$\alpha_s = \tilde{\alpha}_s \circ \mathrm{Ad}\, U(s)$$

$U(s)$ is the solution of

$$\frac{d\,U(s)}{ds} = -iV^{\mathrm{int}}(s)U(s), \quad U(0) = \mathbb{1}$$

and $V^{\mathrm{int}} = \tilde{\alpha}_s^{-1}(V(s))$

Since ω_0 is product and $\tilde{\alpha}_s = \tilde{\alpha}_s^L \otimes \tilde{\alpha}_s^R$, we have

$$\omega_1 = \omega_0 \circ \alpha_1 = \omega_0 \circ \tilde{\alpha}_1 \circ \mathrm{Ad} U_1.$$

In other words, we have states ω_1^L and ω_1^R of the half-chains such that

$$\omega_1 = \omega_1^L \otimes \omega_1^R \circ \mathrm{Ad} U_1 \tag{1}$$

In particular $\omega_1 \underset{\mathrm{u}}{\sim} \omega_1^L \otimes \omega_1^R$ and then also $\omega_1 \circ \beta_g \underset{\mathrm{u}}{\sim} (\omega_1^L \otimes \omega_1^R) \circ \beta_g$.

Next, recall $\omega_1 \circ \beta_g = \omega_1$ and $\beta_g = \beta_g^L \otimes \beta_g^R$.

Therefore,

$$(\omega_1^L \otimes \omega_1^R) \circ \beta_g = (\omega_1^L \circ \beta_g^L) \otimes (\omega_1^R \circ \beta_g^R) \underset{\mathrm{u}}{\sim} \omega_1 \underset{\mathrm{u}}{\sim} \omega_1^L \otimes \omega_1^R$$

This gives

$$\omega_1^R \circ \beta_g^R \underset{\mathrm{u}}{\sim} \omega_1^R$$

and also

$$\omega_1^L \otimes (\omega_1^R \circ \beta_g^R) \underset{\mathrm{u}}{\sim} \omega_1^L \otimes \omega_1^R \underset{\mathrm{u}}{\sim} \omega_1$$

The lefthand side is unitarily equivalent to $\omega_1 \circ \beta_g^R$, due to (1).

Conclusion:

$$\omega_1 \circ \beta_g^R \underset{\mathrm{u}}{\sim} \omega_1$$

This means that β_g^R is implemented by a unitary U_g^R in the GNS representation of ω_1:

$$\omega_1 \circ \beta_g^R(A) = \langle \Omega, (U_g^R)^* \pi(A) U_g^R \Omega \rangle.$$

Since π is an irreducible representation and

$$(U_h^R)^*(U_g^R)^* \pi(\cdot) U_g^R U_h^R = \pi \circ \beta_{gh} = (U_{gh}^R)^* \pi(\cdot) U_{gh}^R$$

whence

$$U_{gh}^R (U_h^R)^*(U_g^R)^* \pi(\cdot) U_g^R U_h^R (U_{gh}^R)^* = \pi(\cdot),$$

we must gave $c(g,h) \in U(1)$ s.t.

$$U_g^R U_h^R (U_{gh}^R)^* = c(g,h)\mathbb{1}$$

with c belonging to an equivalence class of 2-cycles labeled by an element of $H^2(G, U(1))$.

Second cohomology group $H^2(G, U(1))$ of a finite group.

Let G be a finite group, $\mathcal{H}$ a complex Hilbert space, and $U : G \to \mathcal{B}(\mathcal{H})$ such that

$$U(g)U(g) = c(g,h)U(gh),$$

with $c(g,h) \in U(1)$. U is called a projective representation of G.

Given such U and any $\varphi : G \to U(1)$, $\tilde{U}(g) := \varphi(g)U(g)$ also defines projective representation of G. When two projective representations are related in this way, we call them equivalent.

It is possible that there exists φ such that $\tilde{U}$ is a (proper) unitary representation of G.

The associativity of group multiplication implies the 2 co-cyle property of $c(g,h)$: for $g, h, k \in G$

$$c(gh,k)c(g,h)U(ghk) = U(gh)U(k) = U(g)U(hk) = c(g,hk)c(h,k)U(ghk)$$

Hence

$$c(gh,k)c(g,h) = c(g,hk)c(h,k), \quad g,h,k \in G.$$

The equivalence relation for U's becomes an equivalence relation for 2 co-cycles: c and $\tilde{c}$ are equivalent if there exists $\varphi : G \to U(1)$ for which

$$\tilde{c}(g,h) = \frac{\varphi(g)\varphi(h)}{\varphi(gh)} c(g,h), \quad g,h \in G.$$

The set of equivalence classes of 2-co-cycles is an abelian group for pointwise multiplication: $H^2(G, U(1))$.

For example: $H^2(\mathbb{Z}_n, U(1)) = \{0\}, H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$.

The equivalence class of projective representations of G are in 1-1. correspondence with the elements of $H^2(G, U(1))$.

Remarks

- Similar invariants for fermion models and in 2 two dimensions Ogata, Bourne-Ogata, Sopenko
- There are other approaches focussing on states rather than the interactions Kapustin-Sopenko-Wang, ... 2019-22
- Discrete analogue: one replaces the automorphisms by finite depth circuits (many authors)
- MPS and TNS states served as an inspiration for many of the ideas