Interval Decomposition of Persistence Modules and Coarse Graining

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February 8, 2023

Setup

$\mathcal{V} : (\mathbb{R}, \geq_{\mathbb{R}}) \to \mathbf{Vect_K}$

1. $\mathcal{V}$ is semi continuous: $\forall s \in \mathbb{R},\ \exists\ t \geq s;\ \ \alpha_{s,t'}$ is an isomorphism for all $s \leq t' \leq t$
2. $\mathcal{V}$ admits a finite spectral set $Sp(\mathcal{V})$

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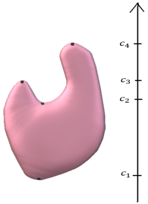
$Sp(\mathcal{V}) :=$

$\{c \in \mathbb{R} \,|\, \exists\ r <_{\mathbb{R}} c\ \ \alpha_{r',c}$ is not an isomorphism $\forall r' \in \mathbb{R}$ with $r \leq_{\mathbb{R}} r' <_{\mathbb{R}} c\}$

$Sp(\mathcal{V}) = \{c_1, c_2, ..., c_e\}$ where $c_1 < c_2 < ... < c_e$



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- $g_{s,t} := \alpha_{s,t}\big|_{G_s^{\mathcal{V}}}$ where $s < t$
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- $K_{c_i,c_k}^{\mathcal{V}}$ as a complement of $Ker(g_{c_i,c_{k-1}}^{\mathcal{V}})$ in $Ker(g_{c_i,c_k}^{\mathcal{V}})$

  $Ker(g_{c_i,c_k}^{\mathcal{V}}) = Ker(g_{c_i,c_{k-1}}^{\mathcal{V}}) \oplus K_{c_i,c_k}^{\mathcal{V}}$

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Claim

$$Bar(\mathcal{V}) = \bigoplus_{(c_i<c_k)}(\mathcal{C}[c_i, c_k))^{n_{c_i,c_k}} \bigoplus(\mathcal{C}[c_i, \infty))^{n_{c_i}}$$

$n_{c_i,c_k} = dim[K^{\mathcal{V}}_{c_i,c_k}]$ and $n_{c_i} = dim[A^{\mathcal{V}}_{c_i}]$

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Theorem

Following statements are equivalent:

(I) $\mathcal{V} \simeq \mathcal{W}$

(II) $Sp(\mathcal{V}) = Sp(\mathcal{W})$ and $A^{\mathcal{V}}_{c_i} \simeq A^{\mathcal{W}}_{c_i}$ and $K^{\mathcal{V}}_{c_i,c_k} \simeq K^{\mathcal{W}}_{c_i,c_k}$, $\forall c_i < c_k \in Sp(\mathcal{V})$

$(I) \implies (II)$: easy !

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pointwise decomposition

$\mathcal{V}(s) = \bigoplus_{c_k > s} K^{\mathcal{V}}_{s,c_k} \bigoplus A^{\mathcal{V}}_s \bigoplus P^{\mathcal{V}}_s$

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$$\mathcal{V}(s) = G^{\mathcal{V}}_s \bigoplus P^{\mathcal{V}}_s = Ker(g^{\mathcal{V}}_{s,c_e}) \bigoplus A^{\mathcal{V}}_s \bigoplus P^{\mathcal{V}}_s = \bigoplus_{c_k > s} K^{\mathcal{V}}_{s,c_k} \bigoplus A^{\mathcal{V}}_s \bigoplus P^{\mathcal{V}}_s$$

lemma

$P^{\mathcal{V}}_s = \bigoplus_{c_i < s} Im(g^{\mathcal{V}}_{s,c_i})$

since $Im(g^{\mathcal{V}}_{c_j,s}) \cap Im(g^{\mathcal{V}}_{c_k,s}) = 0_{\mathcal{V}(s)}$

lemma

$P_s^{\mathcal{V}} \simeq \bigoplus_{c_i < s} A_{c_i}^{\mathcal{V}} \bigoplus_{c_i < s \leq c_k} K_{c_i, c_k}^{\mathcal{V}}$

$(II) \implies (I)$

$Sp(\mathcal{V}) = Sp(\mathcal{W})$ and $A^{\mathcal{V}}_{c_i} \simeq A^{\mathcal{W}}_{c_i}$ and $K^{\mathcal{V}}_{c_i,c_k} \simeq K^{\mathcal{W}}_{c_i,c_k}$, $\forall c_i < c_k \in Sp(\mathcal{V})$.

Goal:
Existence of two natural transformations $\phi : \mathcal{V} \to \mathcal{W}$ and $\psi : \mathcal{W} \to \mathcal{V}$ such that $\phi \circ \psi = id_{\mathcal{W}}$ and $\psi \circ \phi = id_{\mathcal{V}}$.

$(II) \Longrightarrow (I)$

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Existence of two natural transformations
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From the point-wise decomposition :

$$\mathcal{V}(s) = \bigoplus_{c_k > s} K^{\mathcal{V}}_{s,c_k} \bigoplus A^{\mathcal{V}}_s \bigoplus P^{\mathcal{V}}_s \qquad P^{\mathcal{V}}_s \simeq \bigoplus_{c_i < s} A^{\mathcal{V}}_{c_i} \bigoplus_{c_i < s \leq c_k} K^{\mathcal{V}}_{c_i,c_k}$$

$$\mathcal{W}(s) = \bigoplus_{c_k > s} K^{\mathcal{V}}_{s,c_k} \bigoplus A^{\mathcal{W}}_s \bigoplus P^{\mathcal{V}}_s \qquad P^{\mathcal{W}}_s \simeq \bigoplus_{c_i < s} A^{\mathcal{W}}_{c_i} \bigoplus_{c_i < s \leq c_k} K^{\mathcal{W}}_{c_i,c_k}$$

$\Longrightarrow$

Exsitence of the family of isomorphisms

$$\begin{array}{ccc} \mathcal{V}(s) & \xrightarrow{\alpha_{s,t}} & \mathcal{V}(t) \\ \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_t} \\ \mathcal{W}(s) & \xrightarrow{\beta_{s,t}} & \mathcal{W}(t) \end{array}$$

$$\mathcal{V}(s) = A_s^{\mathcal{V}} \bigoplus_{s<c_k} K_{s,c_k}^{\mathcal{V}} \bigoplus_{c_i<s} Im(g_{c_i,s}^{\mathcal{V}})$$

$$\mathcal{W}(s) = A_s^{\mathcal{W}} \bigoplus_{s<c_k} K_{s,c_k}^{\mathcal{W}} \bigoplus_{c_i<s} Im(g_{c_i,s}^{\mathcal{W}})$$

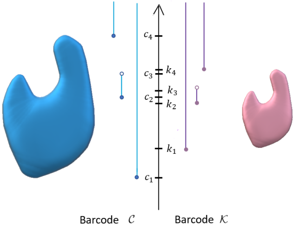
Exsitence of the family of isomorphisms $\{\phi_s\}_{s\in\mathbb{R}}$ such that :

$$\begin{array}{ccc} \mathcal{V}(s) & \xrightarrow{\alpha_{s,t}} & \mathcal{V}(t) \\ \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_t} \\ \mathcal{W}(s) & \xrightarrow{\beta_{s,t}} & \mathcal{W}(t) \end{array}$$

$$\mathcal{V}(s) = G_s^{\mathcal{V}} \oplus P_s^{\mathcal{V}} = G_s^{\mathcal{V}} \oplus \bigoplus_{c_i<s} Im(g_{c_i,s}^{\mathcal{V}})$$

$$\begin{array}{ccc} Im(g_{c_i,s}^{\mathcal{V}}) & \longrightarrow & Im(g_{c_i,t}^{\mathcal{V}}) \\ \downarrow{\scriptstyle \Lambda_{c_i,s}} & & \downarrow{\scriptstyle \Lambda_{c_i,t}} \\ Im(g_{c_i,s}^{\mathcal{W}}) & \longrightarrow & Im(g_{c_i,t}^{\mathcal{W}}) \end{array} \qquad \begin{array}{ccc} G_{c_i}^{\mathcal{V}} & \longrightarrow & Im(g_{c_i,s}^{\mathcal{V}}) \\ \downarrow{\scriptstyle \phi_{c_i}|_{G_{c_i}^{\mathcal{V}}}} & & \downarrow{\scriptstyle \Lambda_{c_i,s}} \\ G_{c_i}^{\mathcal{W}} & \longrightarrow & Im(g_{c_i,s}^{\mathcal{W}}) \end{array}$$

Coarse graining



$$\lceil r \rceil^{\theta} := min \, \{s \in \{m\theta\}_{m \in \mathbb{N}^0} \mid \; r \leq s\}$$

$$\lfloor r \rfloor_{\theta} := max \, \{s \in \{m\theta\}_{m \in \mathbb{N}^0} \mid \; r \geq s\}$$

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Lemma

$$\lceil r \rceil^{\theta} \geq \lfloor t \rfloor_{\theta} \quad \Longleftrightarrow \quad t - r < \theta$$

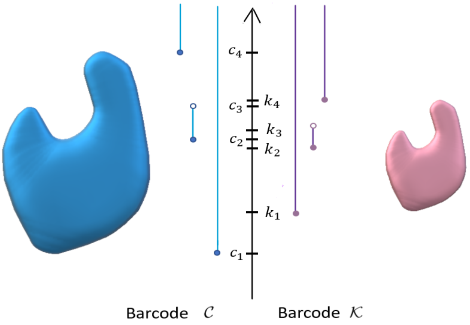
$$\mathcal{B} = \bigoplus_{(c_i < c_k)} (\mathcal{C}[c_i, c_k))^{n_{c_i,c_k}} \bigoplus (\mathcal{C}[c_i, \infty))^{n_{c_i}}$$

θ-coarse grained barcode

$$\widetilde{\mathcal{B}}^{\theta} = \bigoplus_{\lceil c_i \rceil^{\theta} < \lfloor c_k \rfloor_{\theta}} \mathcal{C}[\lceil c_i \rceil^{\theta}, \lfloor c_k \rfloor_{\theta})^{n_{\lceil c_i \rceil^{\theta} \lfloor c_k \rfloor_{\theta}}} \bigoplus_{\lceil c_i \rceil^{\theta}} \mathcal{C}[\lceil c_i \rceil^{\theta}, \infty)^{n_{\lceil c_i \rceil^{\theta}}}$$

$$\mathrm{d}_{bott}(\widetilde{\mathcal{B}}^{\theta}, \mathcal{B}) \leq \theta$$

Outlook



$$\widetilde{\mathcal{B}}^{\theta} = \bigoplus_{\lceil c_i \rceil^{\theta} < \lfloor c_k \rfloor_{\theta}} \mathcal{C}[\lceil c_i \rceil^{\theta}, \lfloor c_k \rfloor_{\theta})^{n_{\lceil c_i \rceil^{\theta} \lfloor c_k \rfloor_{\theta}}} \bigoplus_{\lceil c_i \rceil^{\theta}} \mathcal{C}[\lceil c_i \rceil^{\theta}, \infty)^{n_{\lceil c_i \rceil^{\theta}}}$$

To get the idea of the proof:

$Sp(\mathcal{V}) = \{c_1, c_2, c_3\}$

Recipe !

$$A_{c_1}^{\mathcal{V}} \oplus K_{c_1,c_2}^{\mathcal{V}} \oplus K_{c_1,c_3}^{\mathcal{V}} \longrightarrow A_{c_2}^{\mathcal{V}} \oplus K_{c_2,c_3}^{\mathcal{V}} \oplus Im(g_{c_1,c_2}^{\mathcal{V}}) \longrightarrow A_{c_3}^{\mathcal{V}} \oplus Im(g_{c_1,c_3}^{\mathcal{V}}) \oplus Im(g_{c_2,c_3}^{\mathcal{V}})$$

$$A_{c_1}^{\mathcal{W}} \oplus K_{c_1,c_2}^{\mathcal{W}} \oplus K_{c_1,c_3}^{\mathcal{W}} \longrightarrow A_{c_2}^{\mathcal{W}} \oplus K_{c_2,c_3}^{\mathcal{W}} \oplus Im(g_{c_1,c_2}^{\mathcal{W}}) \longrightarrow A_{c_3}^{\mathcal{W}} \oplus Im(g_{c_1,c_3}^{\mathcal{W}}) \oplus Im(g_{c_2,c_3}^{\mathcal{W}})$$

start with oldest ones

$$\widetilde{\mathcal{B}}^{\theta} = \bigoplus_{\lceil c_i \rceil^{\theta} < \lfloor c_k \rfloor_{\theta}} \mathcal{C}[\lceil c_i \rceil^{\theta}, \lfloor c_k \rfloor_{\theta})^{n_{\lceil c_i \rceil^{\theta} \lfloor c_k \rfloor_{\theta}}} \bigoplus_{\lceil c_i \rceil^{\theta}} \mathcal{C}[\lceil c_i \rceil^{\theta}, \infty)^{n_{\lceil c_i \rceil^{\theta}}}$$

- **Choose** $\Lambda_{c_1,c_3} : Im(g_{c_1,c_3}^{\mathcal{V}}) \to Im(g_{c_1,c_3}^{\mathcal{W}})$

  Define $\Gamma_{c_1} : A_{c_1}^{\mathcal{V}} \to A_{c_1}^{\mathcal{W}}$

$$\Gamma_{c_j} := (g_{c_j,c_e}^{\mathcal{W}}) \circ \Lambda_{c_j,c_e} \circ (\sigma_{c_j,c_e}^{\mathcal{V}} \quad)^{-1}$$

  where

$$\sigma_{c_j,c_e}^{\mathcal{W}} := g_{c_j,c_e}^{\mathcal{W}}\big|_{A_{c_j}^{\mathcal{W}}}$$

$$A^{\mathcal{V}}_{c_1} \oplus K^{\mathcal{V}}_{c_1,c_2} \oplus K^{\mathcal{V}}_{c_1,c_3} \longrightarrow A^{\mathcal{V}}_{c_2} \oplus K^{\mathcal{V}}_{c_2,c_3} \oplus Im(g^{\mathcal{V}}_{c_1,c_2}) \longrightarrow A^{\mathcal{V}}_{c_3} \oplus Im(g^{\mathcal{V}}_{c_1,c_3}) \oplus Im(g^{\mathcal{V}}_{c_2,c_3})$$

$$A^{\mathcal{W}}_{c_1} \oplus K^{\mathcal{W}}_{c_1,c_2} \oplus K^{\mathcal{W}}_{c_1,c_3} \longrightarrow A^{\mathcal{W}}_{c_2} \oplus K^{\mathcal{W}}_{c_2,c_3} \oplus Im(g^{\mathcal{W}}_{c_1,c_2}) \longrightarrow A^{\mathcal{W}}_{c_3} \oplus Im(g^{\mathcal{W}}_{c_1,c_3}) \oplus Im(g^{\mathcal{W}}_{c_2,c_3})$$

- **Choose** $\Theta_{c_1,c_3} : K^{\mathcal{V}}_{c_1,c_3} \to K^{\mathcal{W}}_{c_1,c_3}$

  Define $\Lambda_{c_1,c_2} : Im(g^{\mathcal{V}}_{c_1,c_2}) \to Im(g^{\mathcal{W}}_{c_1,c_2})$

$$\Lambda_{c_1,c_2} := g^{\mathcal{W}}_{c_1,c_2} \circ (\Gamma_{c_1} \oplus \Theta_{c_1,c_3}) \circ (\sigma^{\mathcal{V}}_{c_1,c_2})^{-1}$$

  where

$$\sigma^{\mathcal{V}}_{c_1,c_2} := g^{\mathcal{V}}_{c_1,c_2}\big|_{A^{\mathcal{V}}_{c_1} \oplus K^{\mathcal{V}}_{c_1,c_3}}$$

lemma

$P_s^{\mathcal{V}} \simeq \bigoplus_{c_i<s} A_{c_i}^{\mathcal{V}} \bigoplus_{c_i<s\leq c_k} K_{c_i,c_k}^{\mathcal{V}}$

Proof.

$$G_{c_i}^{\mathcal{V}} = \bigoplus_{c_i<c_k} K_{c_i,c_k}^{\mathcal{V}} \bigoplus A_{c_i}^{\mathcal{V}} \qquad Im(g_{c_i,s}^{\mathcal{V}}) \simeq \bigoplus_{s<c_k} K_{c_i,c_k}^{\mathcal{V}} \bigoplus A_{c_i}^{\mathcal{V}}$$

$$P_s^{\mathcal{V}} = \bigoplus_{c_i<s} Im(g_{s,c_i}^{\mathcal{V}})$$

$\Longrightarrow$

$$P_s^{\mathcal{V}} \simeq \bigoplus_{c_i<s} A_{c_i}^{\mathcal{V}} \bigoplus_{c_i<s<c_k} K_{c_i,c_k}^{\mathcal{V}}$$

□

- **Choose** $\Lambda_{c_i,c_e} : Im(g^{\mathcal{V}}_{c_i,c_e}) \to Im(g^{\mathcal{W}}_{c_i,c_e})$

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where $\sigma^{\mathcal{V}}_{c_i,c_e} := g^{\mathcal{V}}_{c_i,c_e}\big|_{A^{\mathcal{V}}_{c_i}}$

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- **Define** $\Lambda_{c_i,s} : Im(g^{\mathcal{V}}_{c_i,s}) \to Im(g^{\mathcal{W}}_{c_i,s})$

$$\Lambda_{c_i,s} := g^{\mathcal{W}}_{c_i,s} \circ (\Gamma_{c_i} \oplus_{I(s)} \Theta_{c_i,c_k}) \circ (\sigma^{\mathcal{V}}_{s,c_i})^{-1}$$

where $I(s) := \{k \in \mathbb{N} \,|\, s < c_k\}$