The Localized Union-of-Balls Bifiltration

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Goal

Given a finite point set $P \subseteq \mathbb{R}^d$ and a *center* $q \in \mathbb{R}^d$, we want to study the homology of the union of balls centered in P locally around q.

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We want to do this for varying radii of the balls as well as varying scope of locality.

Outline

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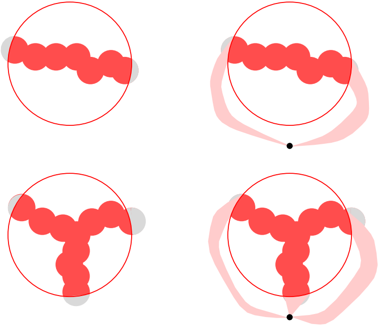

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Two bifiltrations X and Y are *equivalent* whenever there are homotopy equivalences $\phi_{s,r} : X_{s,r} \to Y_{s,r}$ that commute with the inclusion maps.

Part 1: The absolute case

The absolute localized bifiltration

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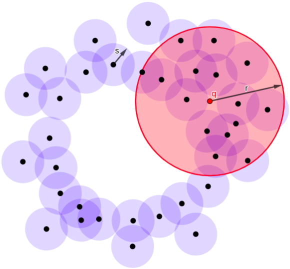

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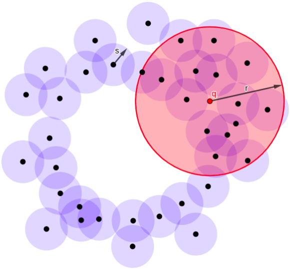

We call $L := (L_{s,r})_{s,r \geq 0}$ the *absolute localized bifiltration*.

The absolute localized bifiltration

The absolute localized bifiltration corresponds to the diagram $\mathbb{R}^2 \to \textbf{Top}$:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & L_{s,r'} & \longrightarrow & L_{s',r'} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
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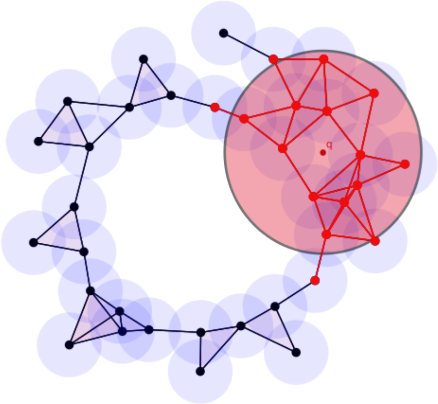
Canonical choice: Restricted Čech filtration. Take the simplicial bifiltration $K := (K_{s,r})_{s,r \geq 0}$ where $K_{s,r} := \mathrm{Nrv}\,\{B_s(p) \cap B_r(q)\}_{p \in P}$



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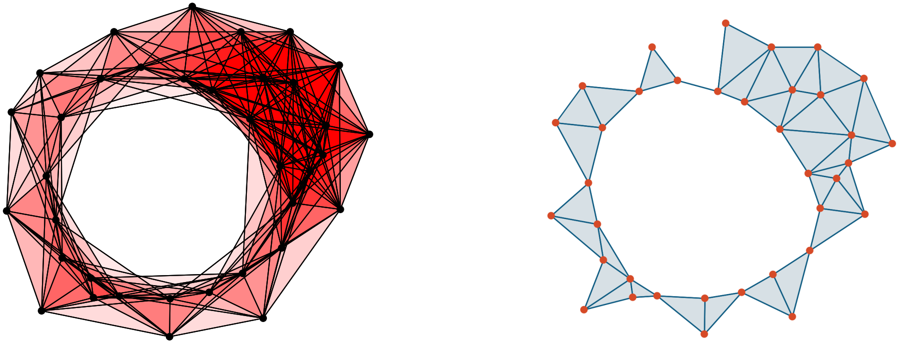

The localized alpha bifiltration

For a point $p \in P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ define its *Voronoi region* as

$$\mathrm{Vor}(p) := \{x \in \mathbb{R}^d \mid \|x - p\| \leq \|x - p'\| \ \forall p' \in P\}$$

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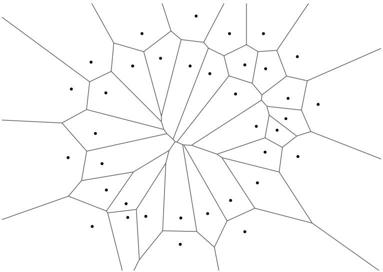
The *Voronoi diagram* of P is the collection of all its Voronoi regions and their boundaries.



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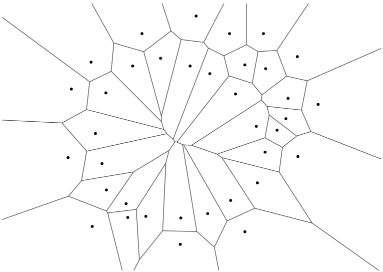
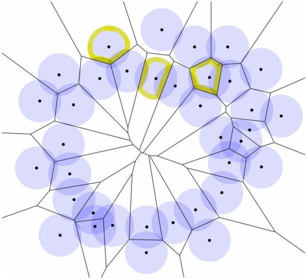


The *Delaunay triangulation* is the nerve of the family of Voronoi regions of P.

The alpha complex

For a point $p \in P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ and radius $s \in \mathbb{R}_{\geq 0}$ we define its *alpha cell* as $R_s(p) := \mathrm{Vor}(p) \cap B_s(p)$.



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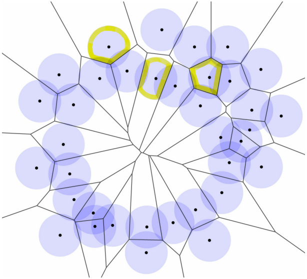

The *alpha complex* is $A_s = \{\{p_1, \ldots, p_k\} \subseteq P \mid R_s(p_1) \cap \cdots \cap R_s(p_k) \neq \emptyset\}$ and the *alpha filtration* is a filtration of alpha complexes.

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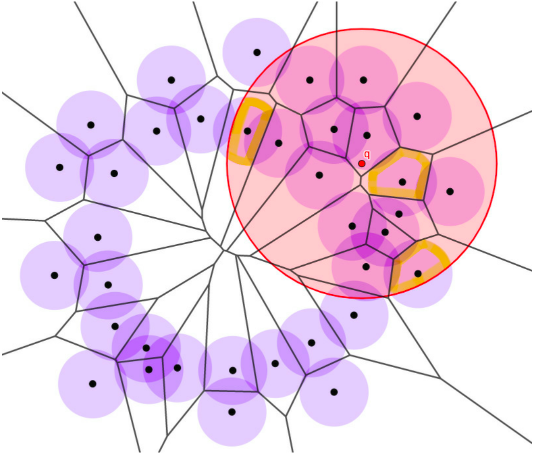

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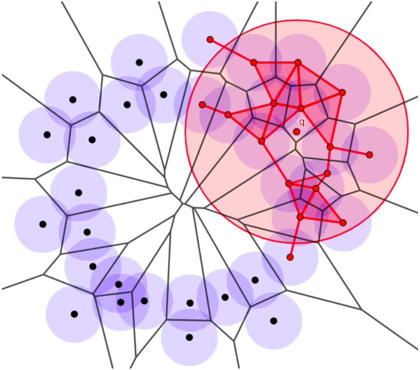
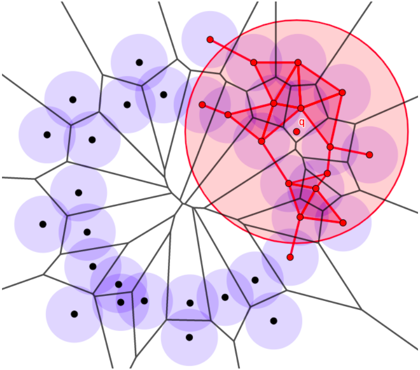


The localized alpha bifiltration

These localized alpha complexes form the *localized alpha bifiltration* $A := \{A_{s,r}\}_{s,r \geq 0}$.



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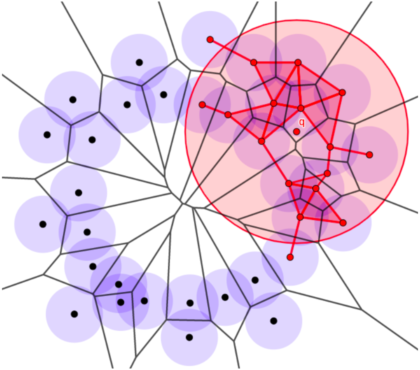
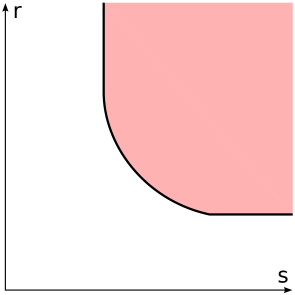
By the persistent nerve theorem, $H_k(L_{s,r}) \cong H_k(A_{s,r})$ and these isomorphisms commute with the inclusion maps of $L_{s,r}$ and $A_{s,r}$. Therefore L and A are *equivalent*



Active regions

We can compute the localized alpha bifiltration $A = \{A_{s,r}\}_{s,r \geq 0}$ if we know for each possible simplex $\sigma \in A$ (that is, each simplex in the Delaunay triangulation) its *active region*

$$R_\sigma = \{(s, r) \in \mathbb{R}^2 \mid \sigma \in A_{s,r}\}$$



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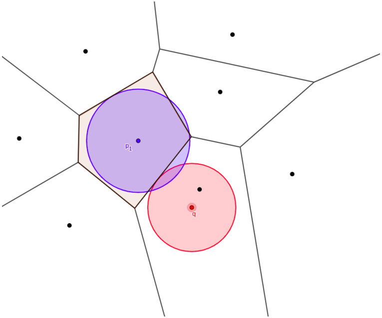
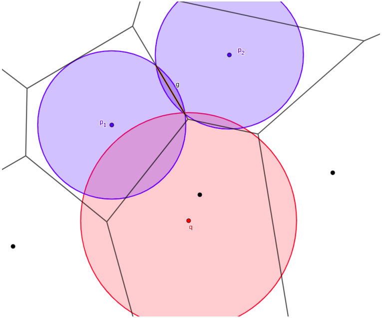
For $\sigma = \{p_1, \ldots, p_k\}$ write $V_\sigma = \mathrm{Vor}(p_1) \cap \cdots \cap \mathrm{Vor}(p_k)$ and $p = p_1$.
Then, $\sigma \in A_{s,r}$ if and only if $V_\sigma \cap B_s(p) \cap B_r(q) \neq \emptyset$.





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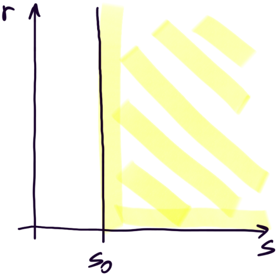

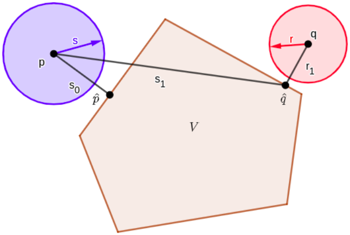

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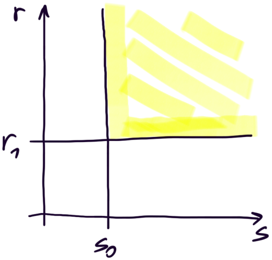

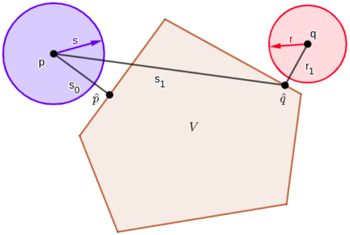

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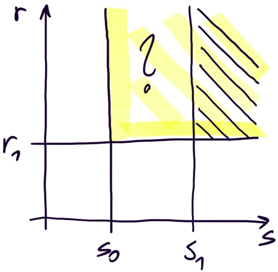
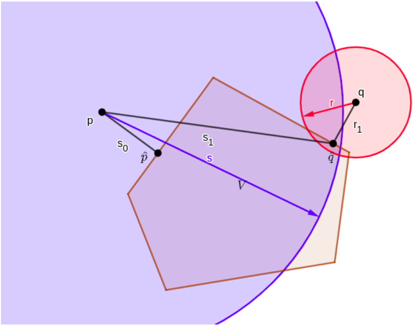
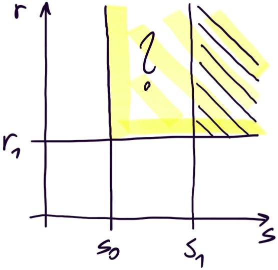
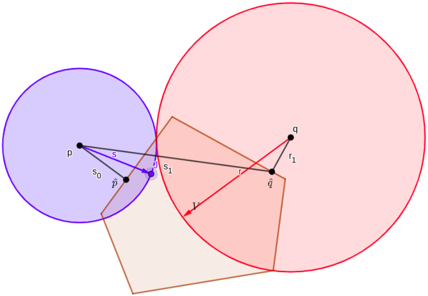




Minimizing paths

We are interested in the boundary of the active region R_σ for $s \in [s_0, s_1]$. This curve is, for $s \in [s_0, s_1]$, the minimal r-value such that $\sigma \in A_{s,r}$ which is the distance of q to the set $V_\sigma \cap B_s(p)$.





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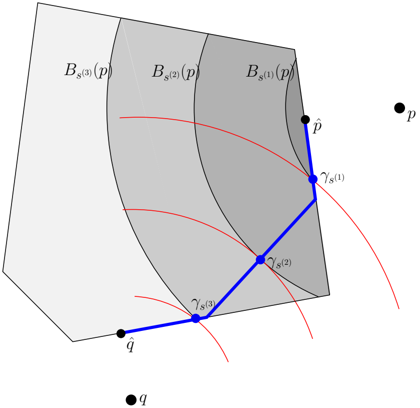

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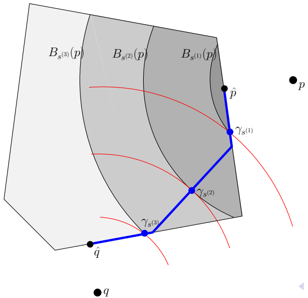

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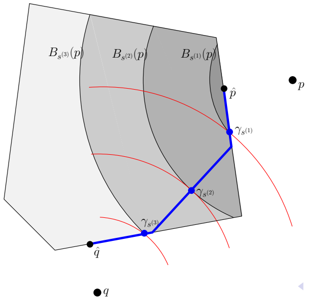

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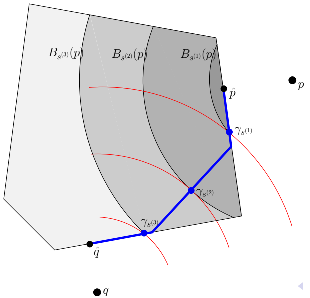

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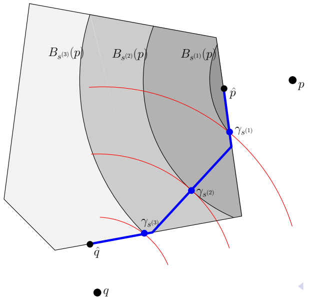
Theorem Every point on the minimizing path lies on a bridge. The minimizing path is a polygonal chain starting in $\hat{p}$ and ending in $\hat{q}$.



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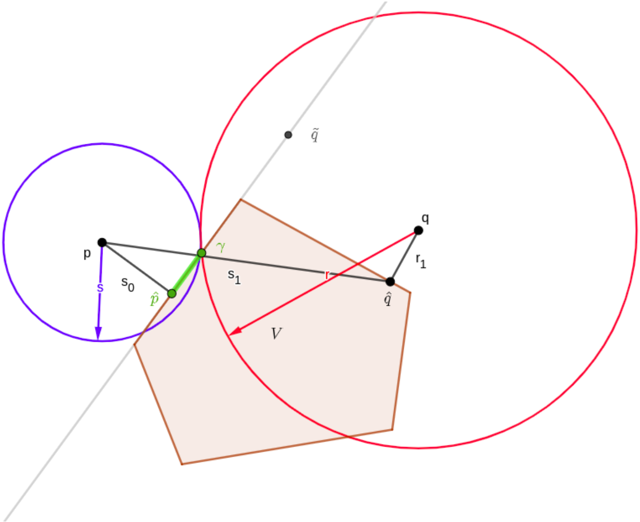

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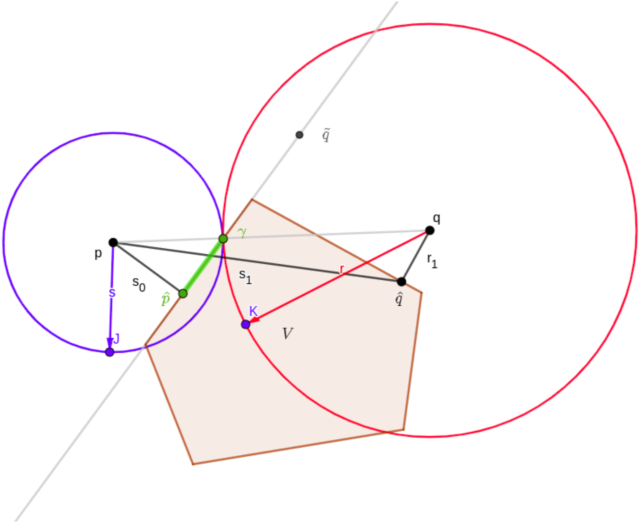

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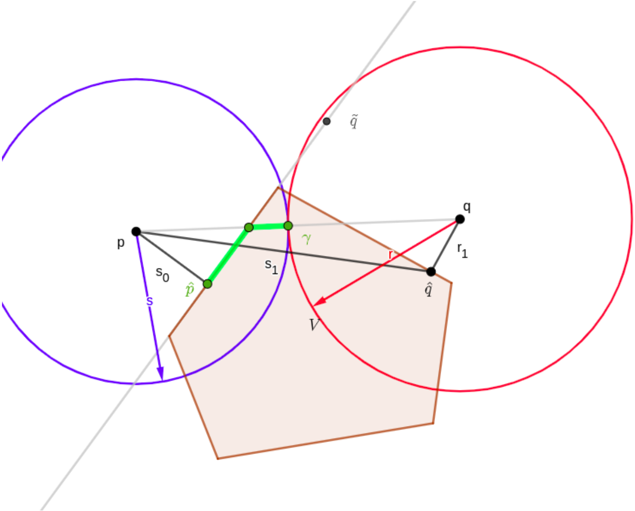

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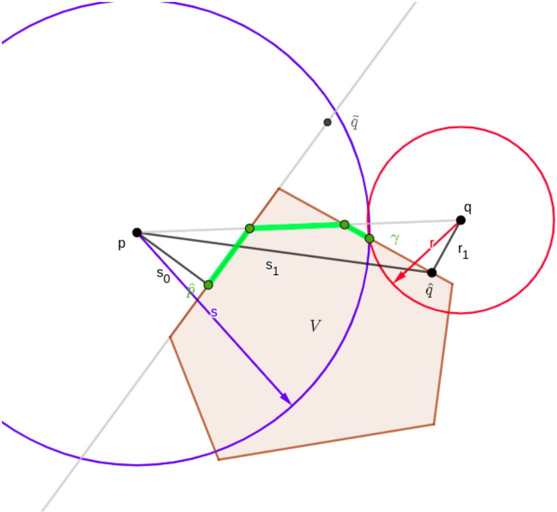

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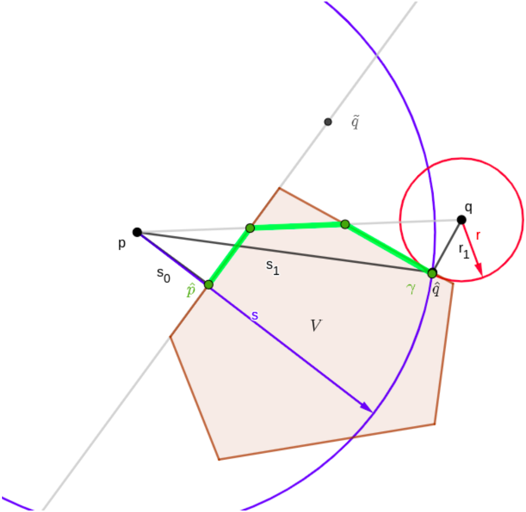

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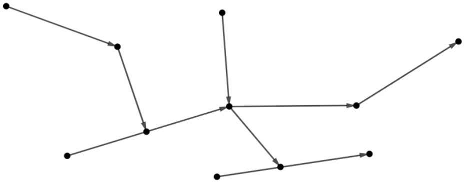

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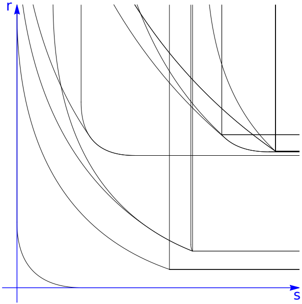
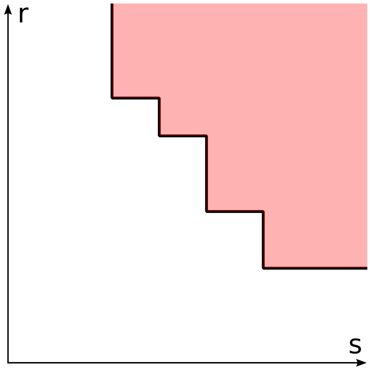
Theorem Let P be n points in general position in $\mathbb{R}^d$ where d is constant. Let N be the size of the Delaunay triangulation of P. We can compute the entry curves of all Delaunay simplices in time $O(N)$.

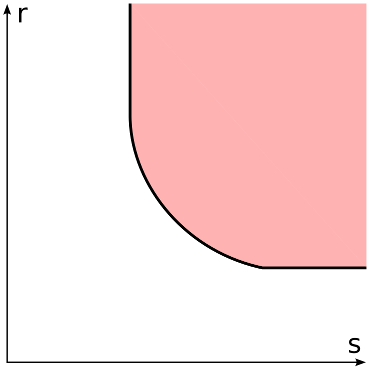
Active regions

For a line segment $\overline{ab}$ of the minimizing path γ, set
$s = \|p - ((1-t)a + tb\|^2$
$r = \|q - ((1-t)a + tb\|^2$
For $t \in [0,1]$ this yields an parabola or line.



∞-criticality





Part 2: The relative case

The relative localized bifiltration

$(s, r) \leq (s', r')$ if $s \leq s'$ and $r \geq r'$.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & (L_s, L_s \backslash B^o_{r'}(q)) & \longrightarrow & (L_{s'}, L_{s'} \backslash B^o_{r'}(q)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & (L_s, L_s \backslash B^o_{r}(q)) & \longrightarrow & (L_{s'}, L_{s'} \backslash B^o_{r}(q)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

The relative localized persistence bimodule

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & H_k(L_s, L_s \backslash B^o_{r'}(q)) & \longrightarrow & H_k(L_{s'}, L_{s'} \backslash B^o_{r'}(q)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & H_k(L_s, L_s \backslash B^o_{r}(q)) & \longrightarrow & H_k(L_{s'}, L_{s'} \backslash B^o_{r}(q)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

The relative localized persistence bimodule

The rows correspond (by excision) to the persistence module

$$\cdots \to H_k(L_s \cap B_r(q), L_s \cap \partial B_r(q)) \to H_k(L_{s'} \cap B_r(q), L_{s'} \cap \partial B_r(q)) \to \cdots$$

which was studied in

- Paul Bendich, David Cohen-Steiner, Herbert Edelsbrunner, John Harer, and Dmitriy Morozov. Inferring Local Homology from Sampled Stratified Spaces. (FOCS 2007)
- Paul Bendich, Bei Wang, and Sayan Mukherjee. Local Homology Transfer and Stratification Learning. (SODA 2012)
- Primoz Skraba and Bei Wang. Approximating Local Homology from Samples. (SODA 2014)

Computing the relative localized persistence bimodule

(X, A) pair of topological spaces.
$\mathcal{U}_X$ cover of X and $\mathcal{U}_A$ cover of A induced by restricting the elements of $\mathcal{U}_X$ to A.

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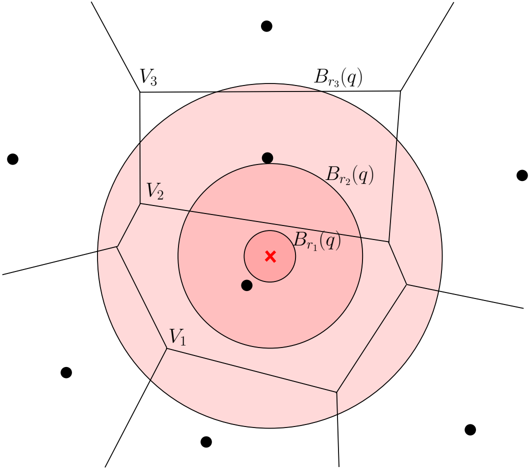
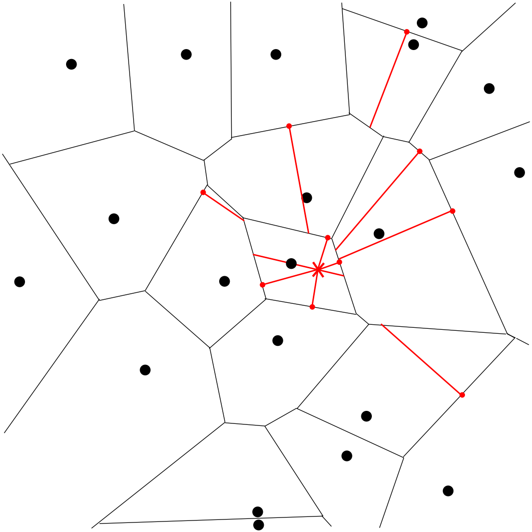
In that case there is a homotopy equivalence $\phi : X \to \mathrm{Nrv}\, \mathcal{U}_X$ which restricts on A to a homotopy equivalence $A \to \mathrm{Nrv}\, \mathcal{U}_X$.

A good cover in $\mathbb{R}^2$



A good cover in $\mathbb{R}^2$



Further work

- Subdivision scheme in relative case for higher dimensions?
- Computation of minimal presentations for ∞-critical bifiltrations?
- Consider a sample of centers and analyse ensemble of localized bifiltrations?
- Applications?