Chromatic alpha complexes

Ondřej Draganov

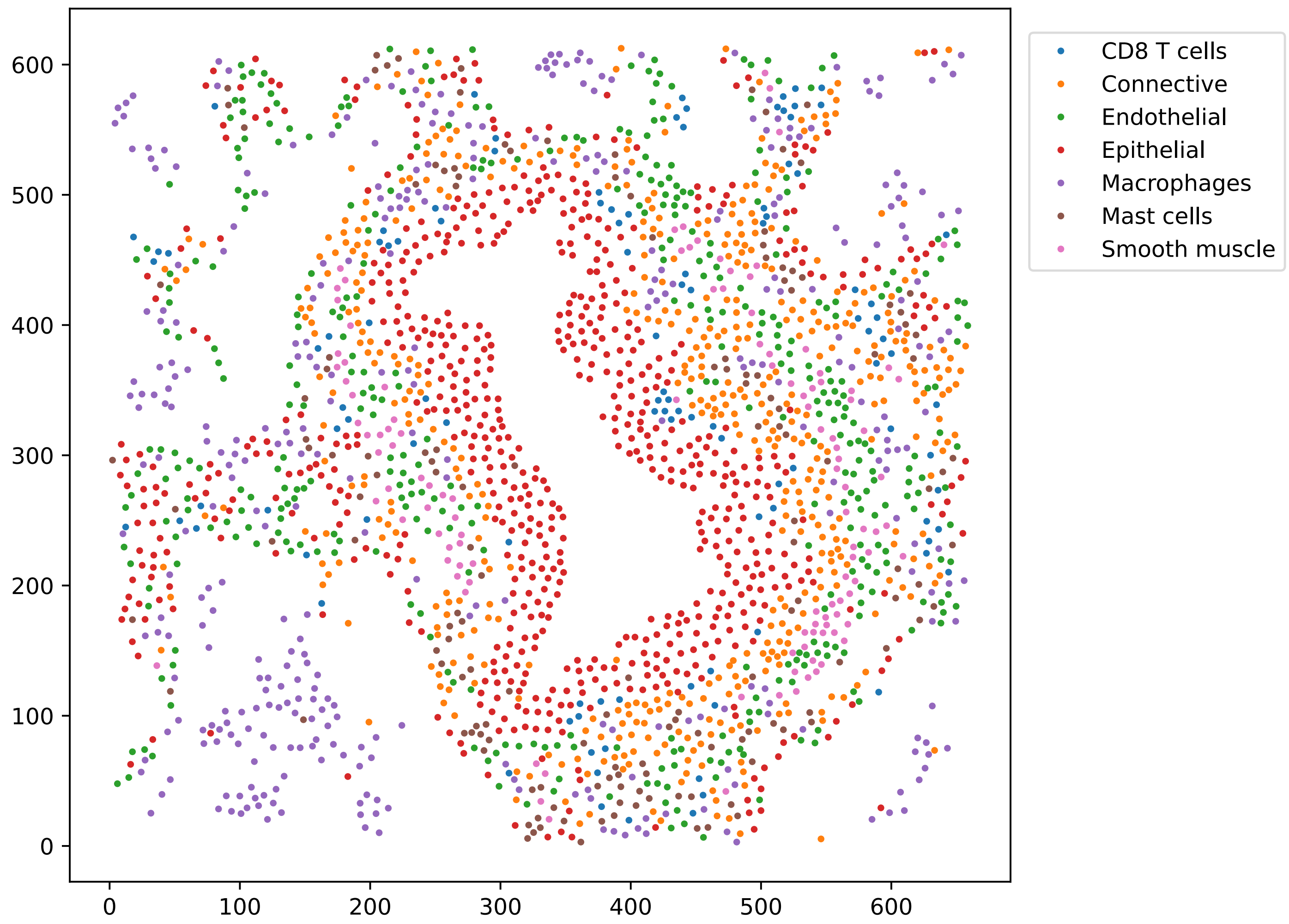
Talk at TU Munich, 18. 7. 2023

Joint work with S. Cultrera di Montesano,

H. Edelsbrunner, and M. Saghafian

Institute of Science and Technology Austria

Motivation comes from spatial biology



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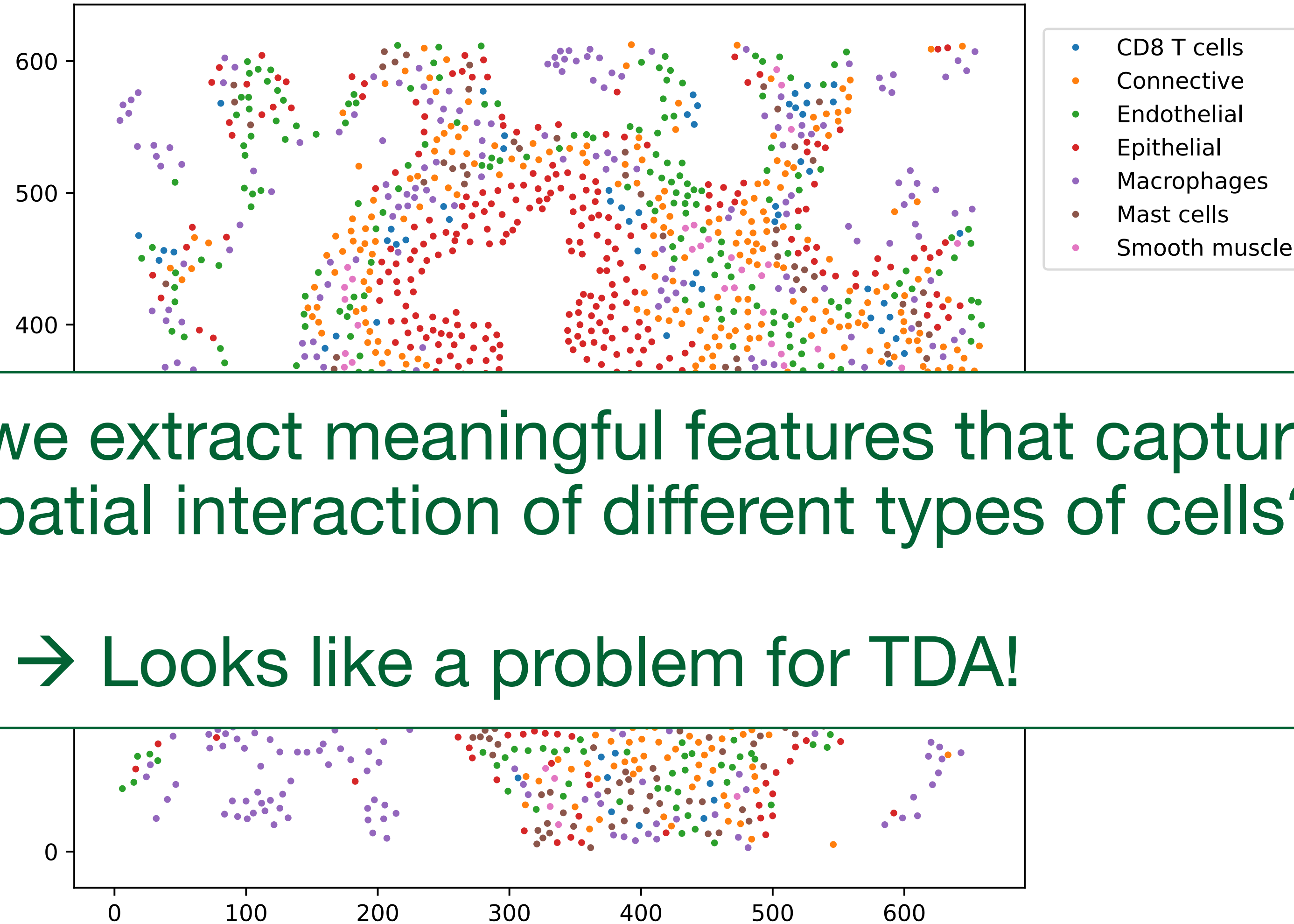
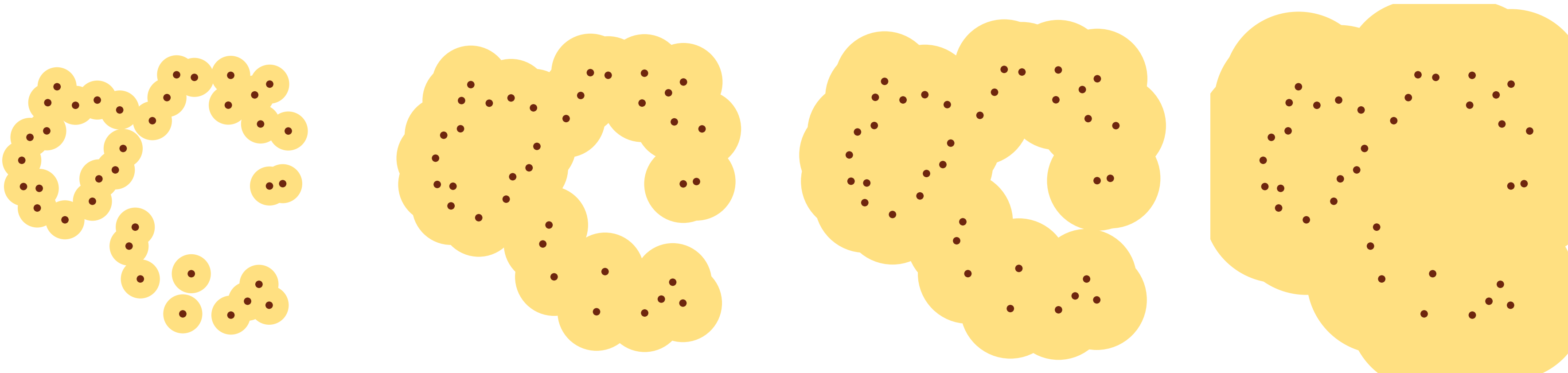


Can we extract meaningful features that capture the spatial interaction of different types of cells?

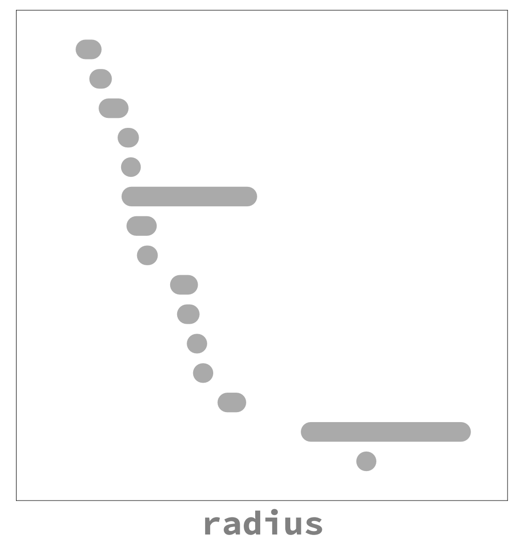
→ Looks like a problem for TDA!

Standard persistent homology pipeline



$$\cdots \longrightarrow \bigcup_{a \in A} B_r(a) \longrightarrow \bigcup_{a \in A} B_{r'}(a) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{H}_p\Big(\bigcup_{a \in A} B_r(a)\Big) \longrightarrow \mathrm{H}_p\Big(\bigcup_{a \in A} B_{r'}(a)\Big) \longrightarrow \cdots$$



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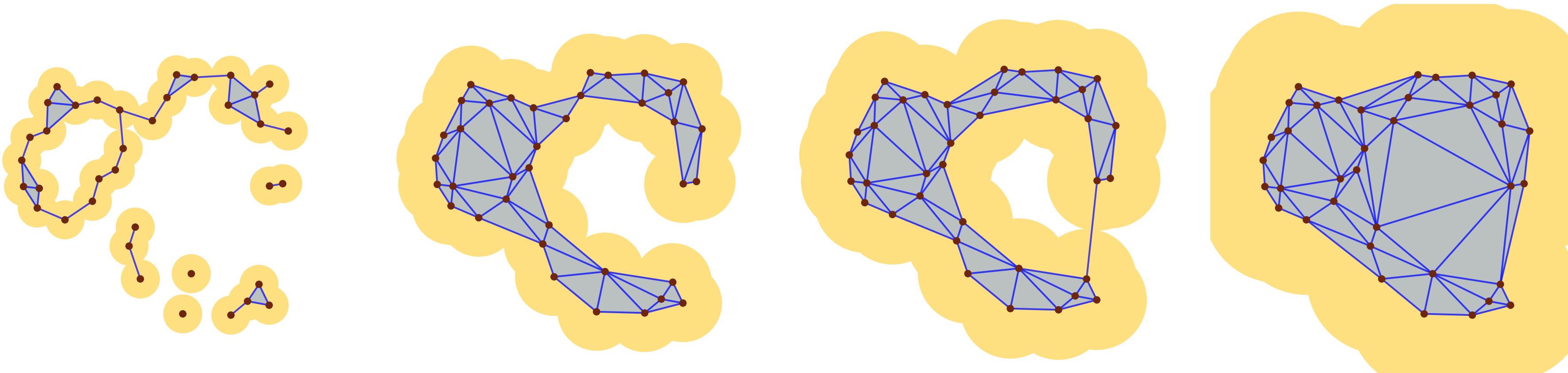
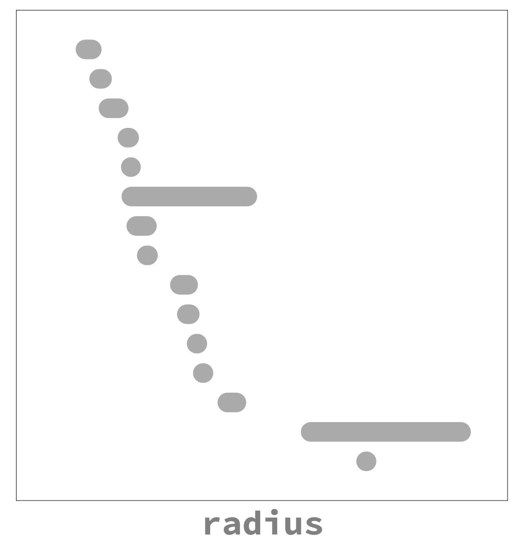
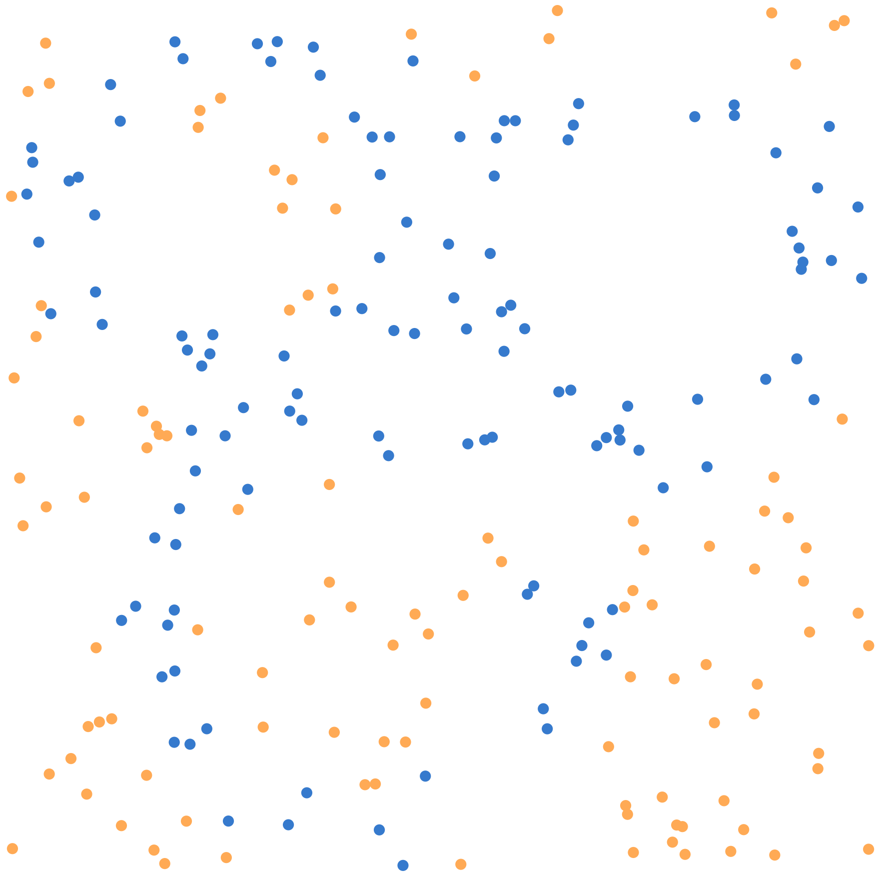


$$\cdots \longrightarrow \mathrm{Alf}_r(A) \longrightarrow \mathrm{Alf}_{r'}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{H}_p\Big(\mathrm{Alf}_r(A)\Big) \longrightarrow \mathrm{H}_p\Big(\mathrm{Alf}_{r'}(A)\Big) \longrightarrow \cdots$$



What can we do with colored points?



- Can we capture something about the *interaction* of the two colors?

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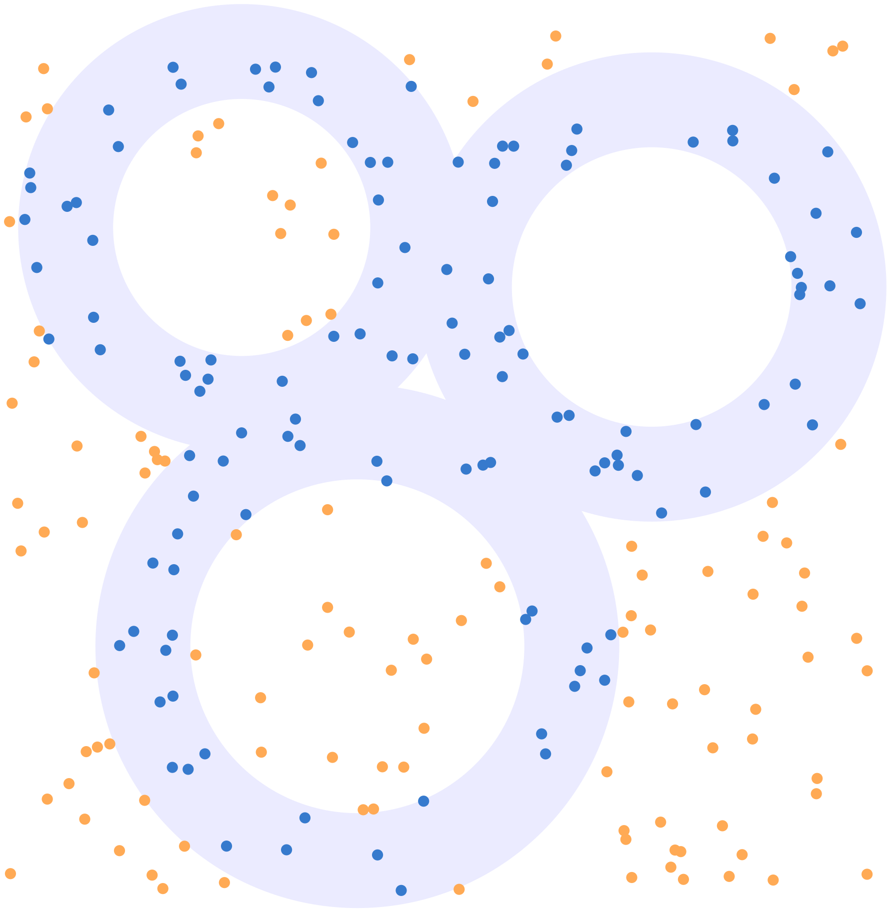

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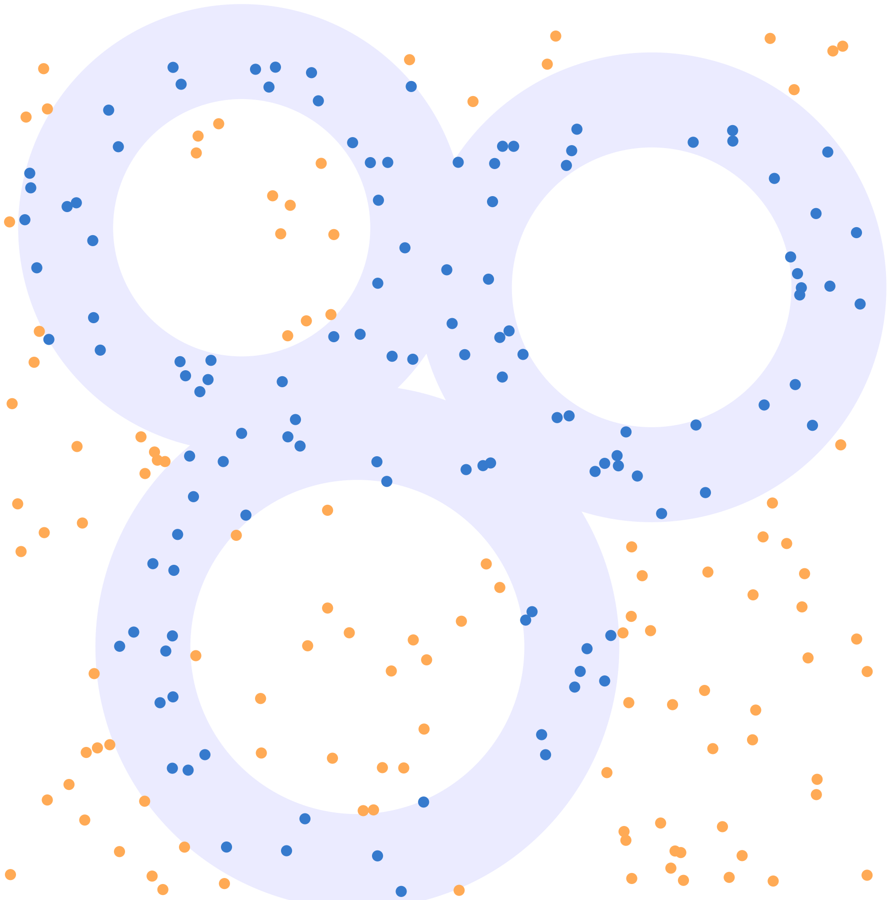

- Can we capture something about the *interaction* of the two colors?
- e.g., blue loops **filled** by orange points

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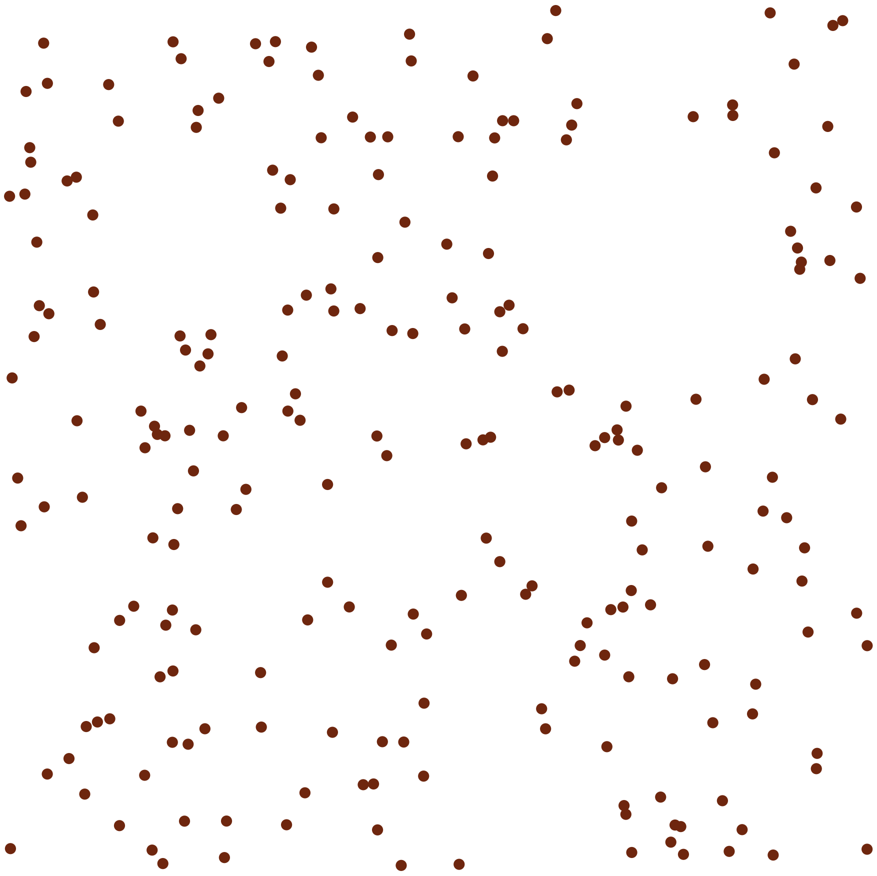

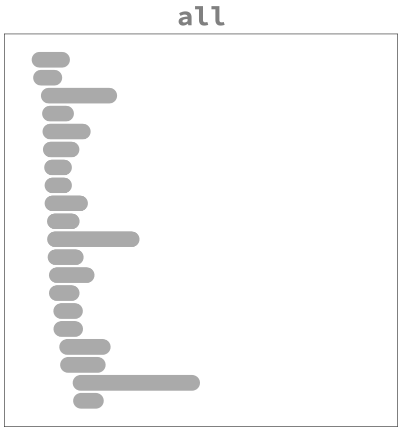

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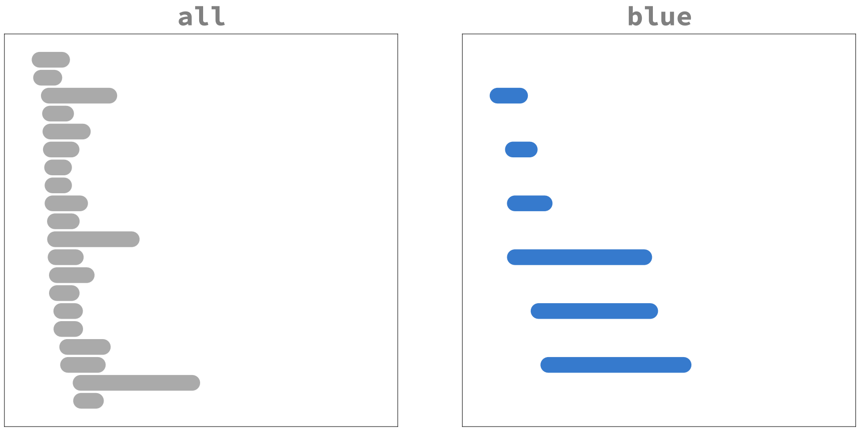

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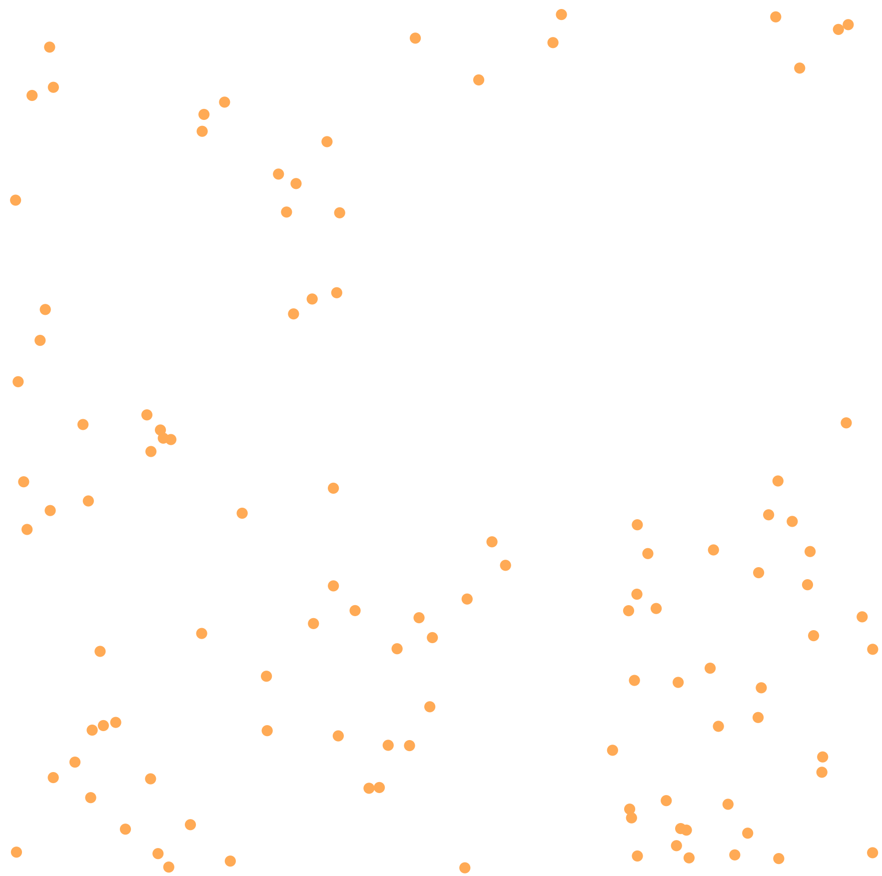

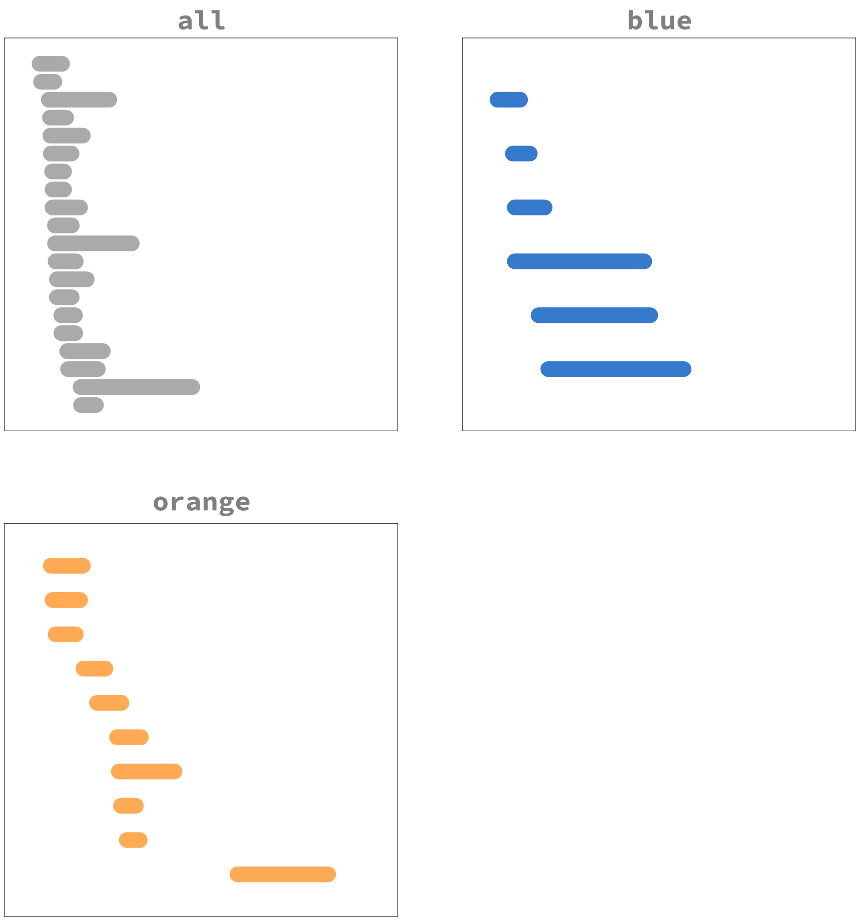

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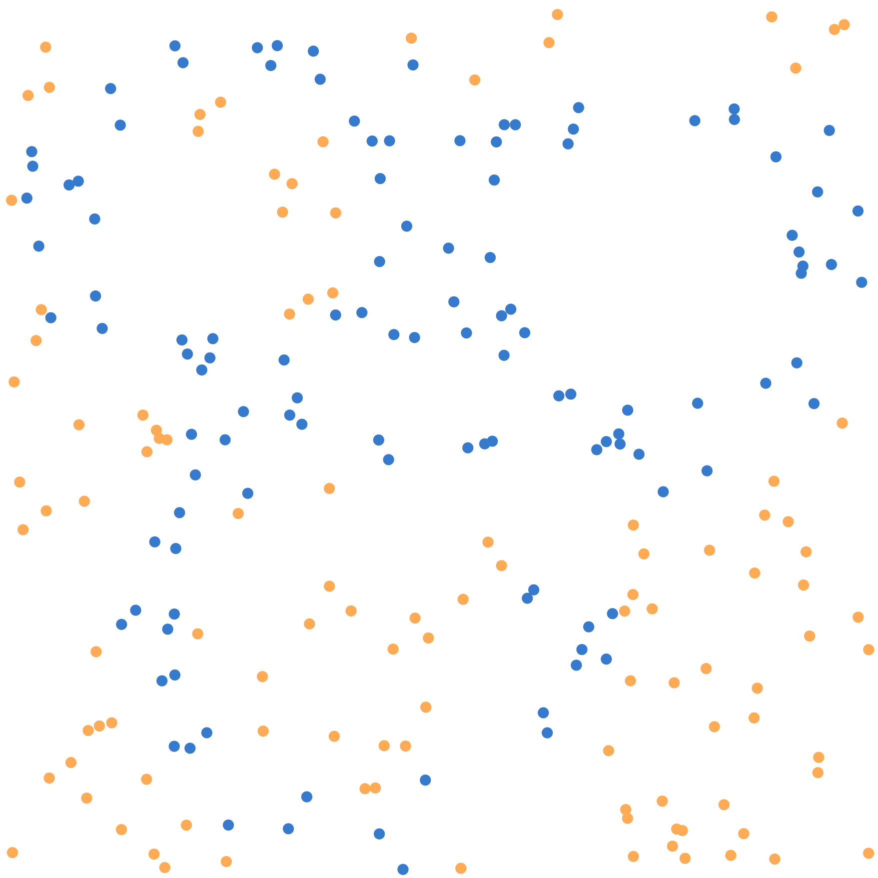

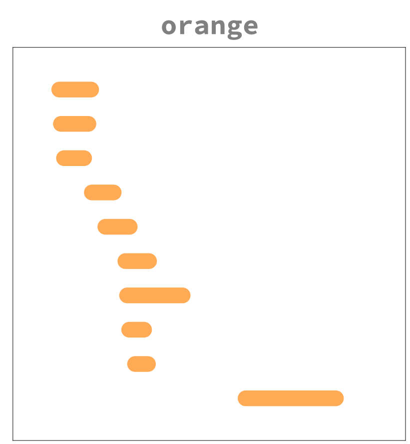

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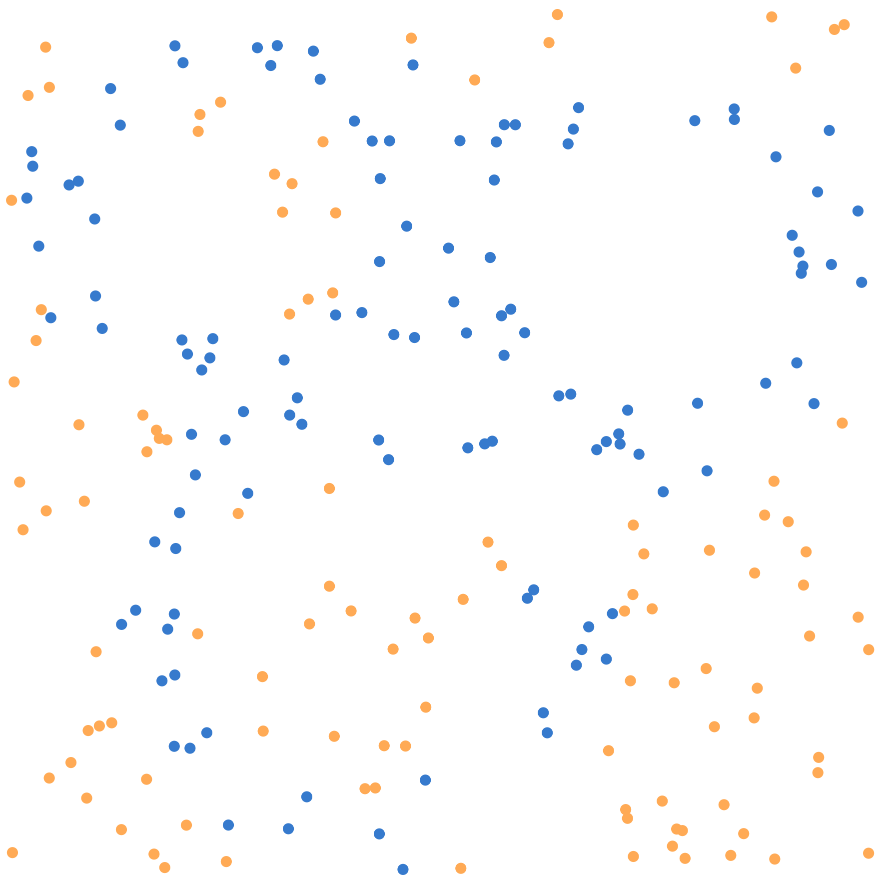
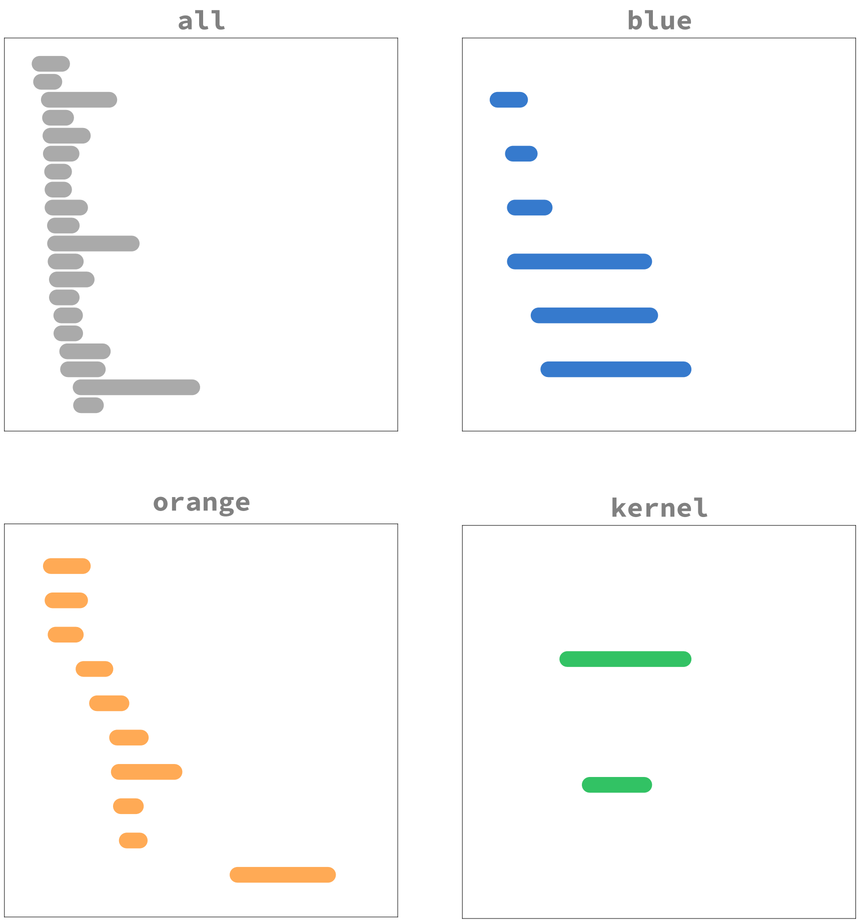
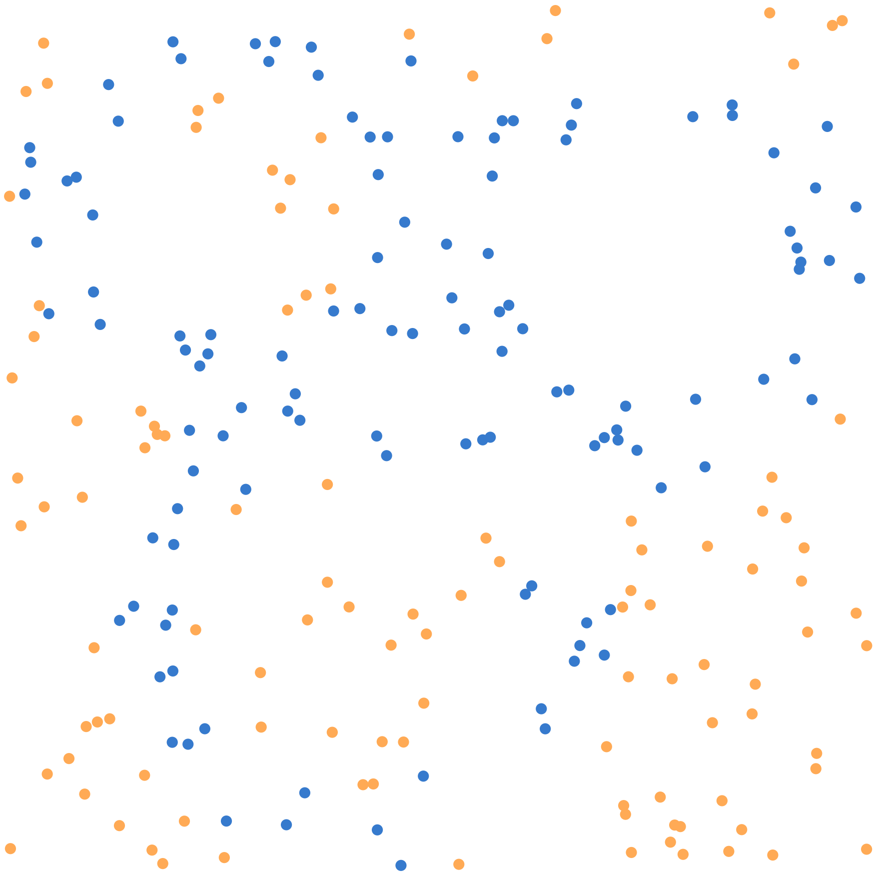
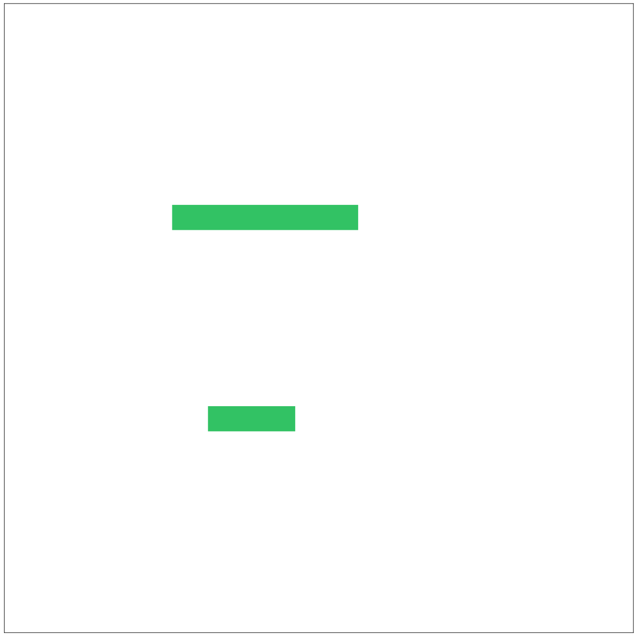




What are those “kernel” features?



kernel



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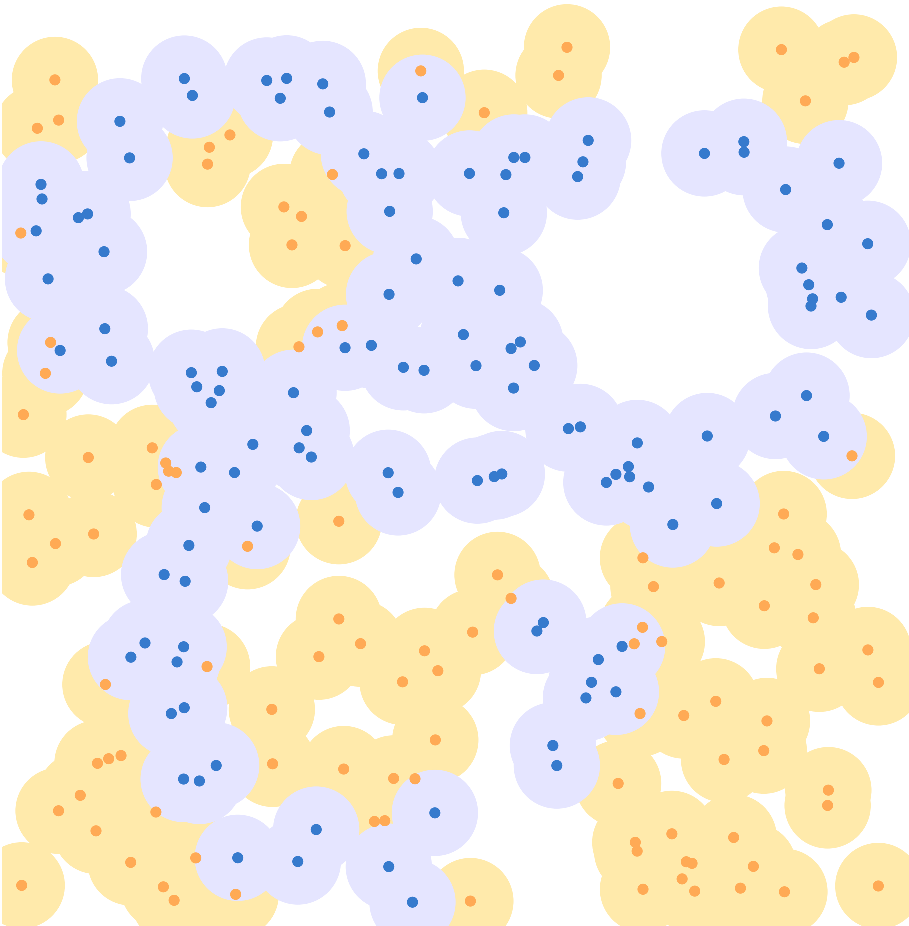
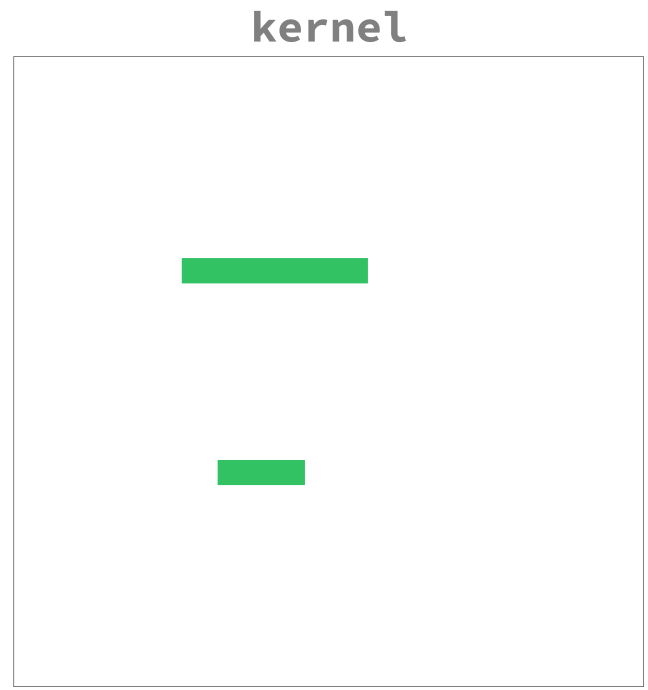


Study the union of blue disks included into the union of blue and orange disks



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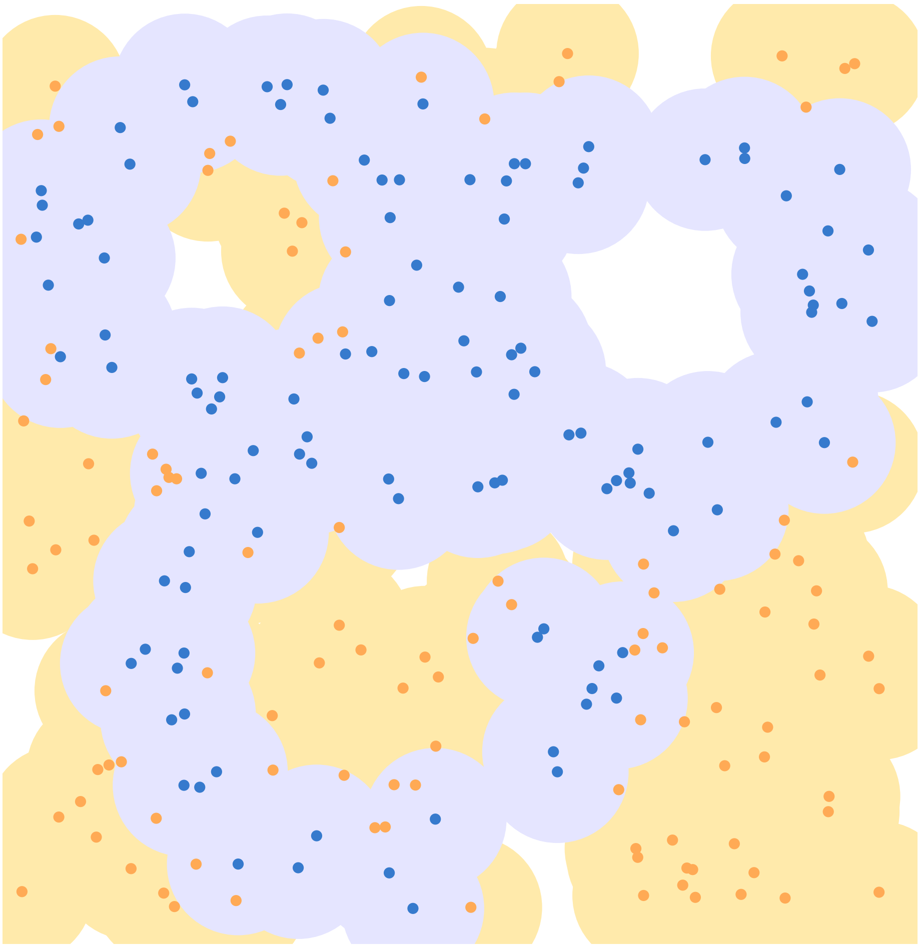

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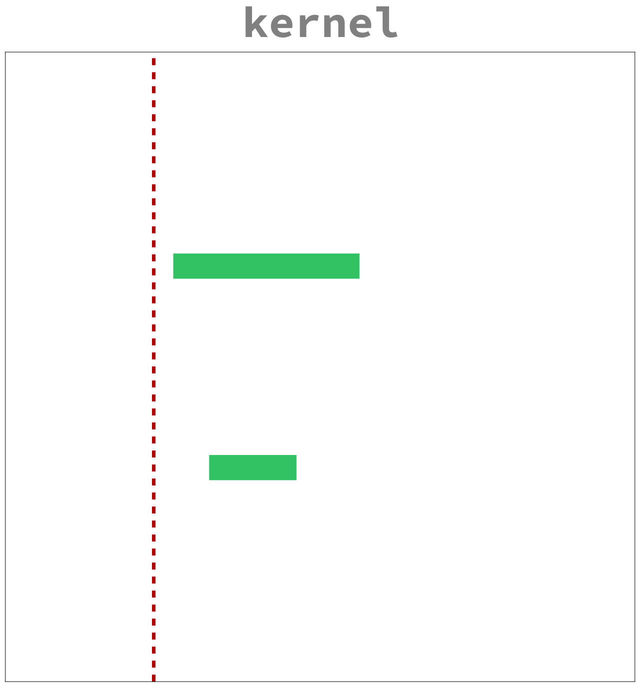

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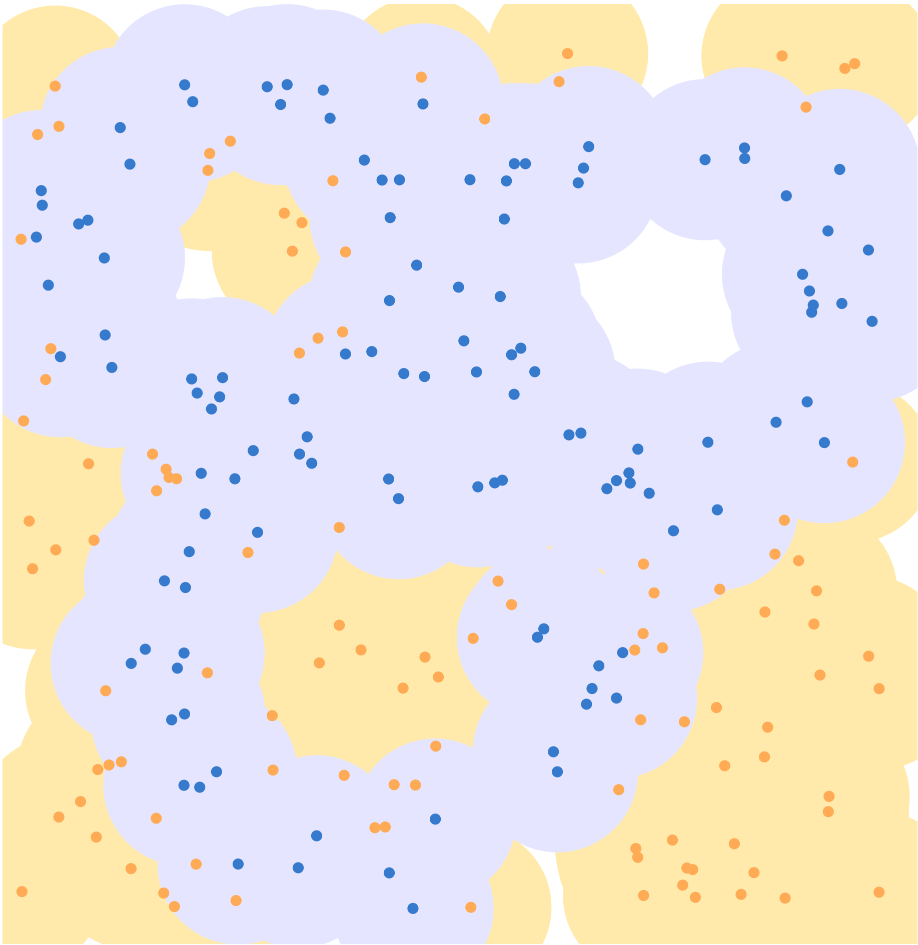

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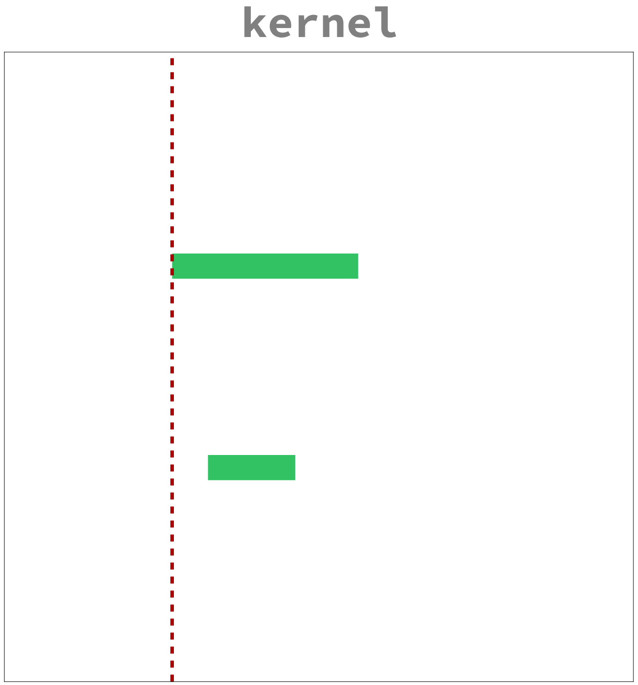

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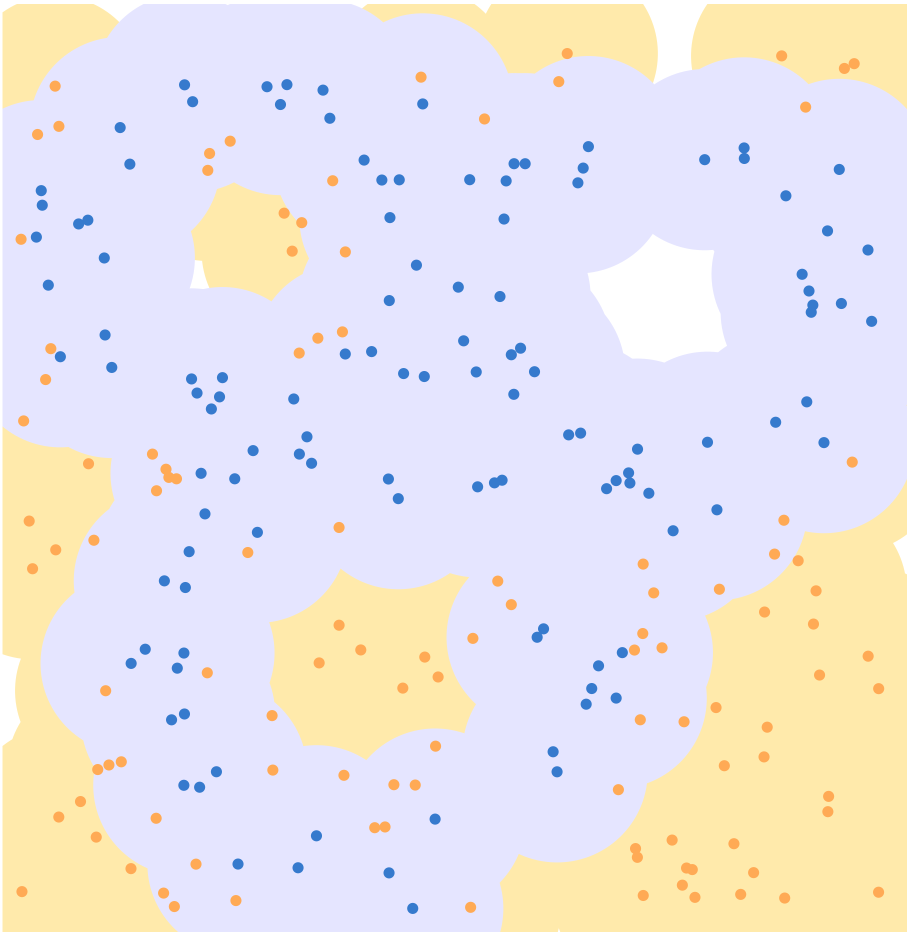

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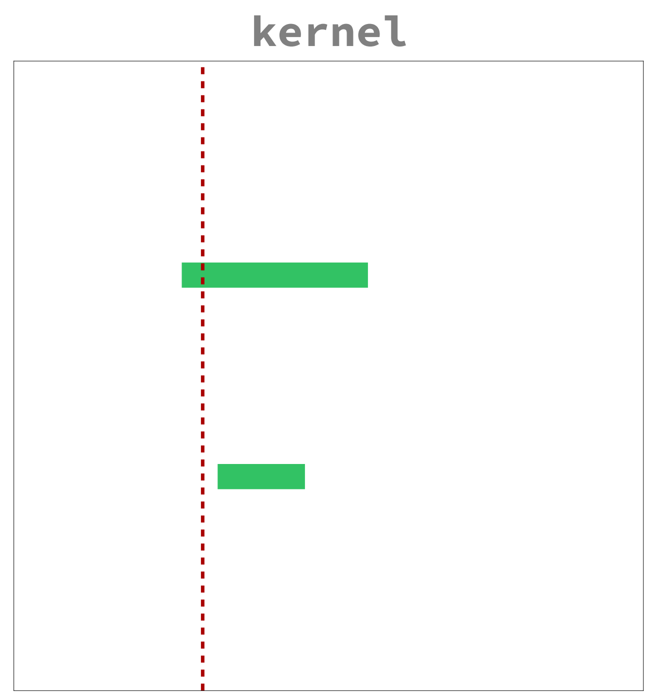

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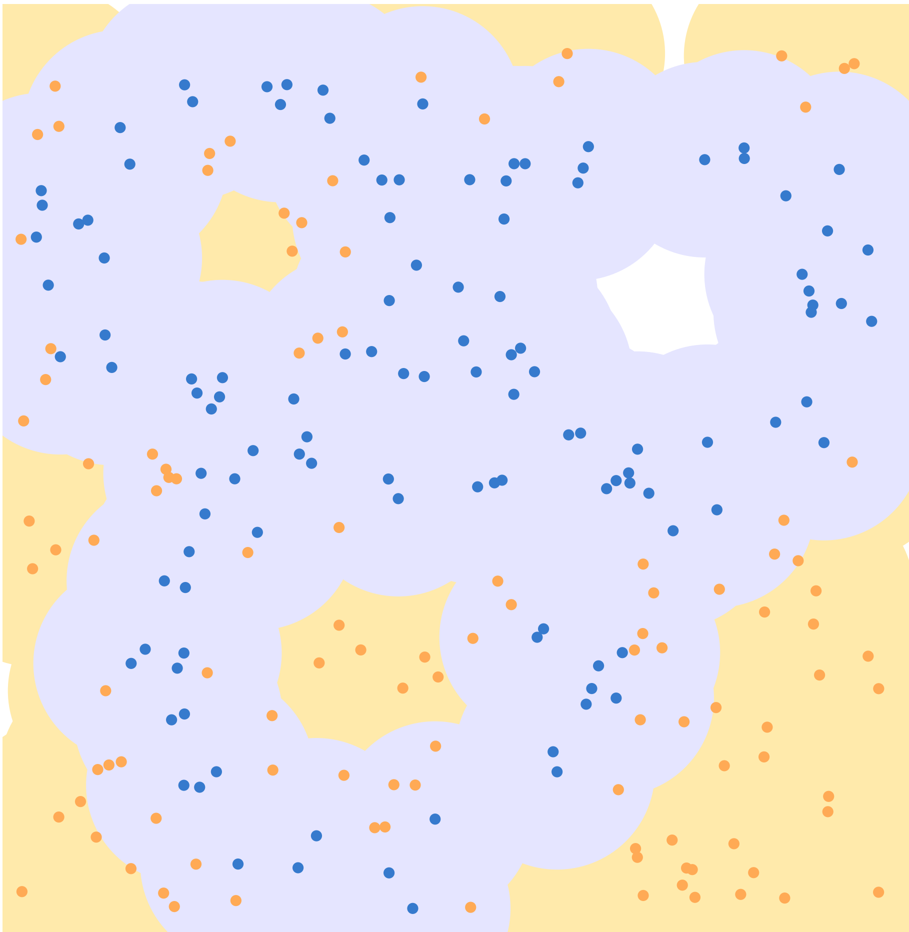

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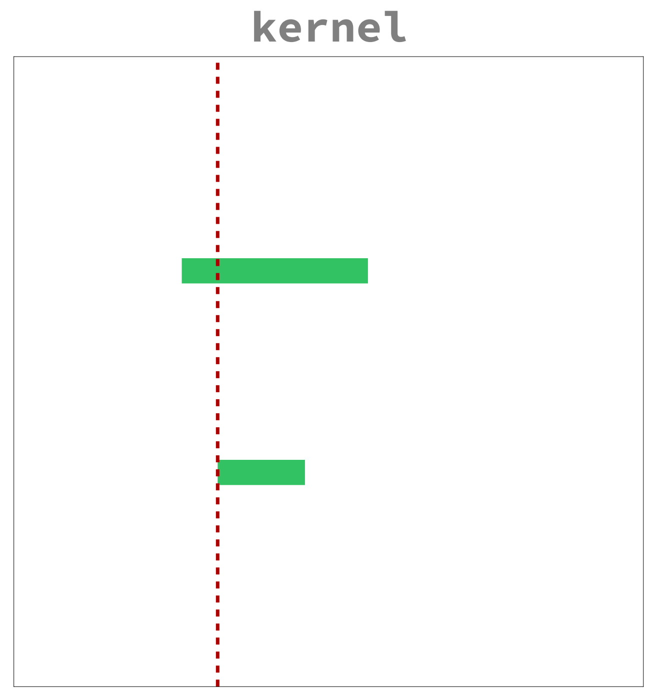

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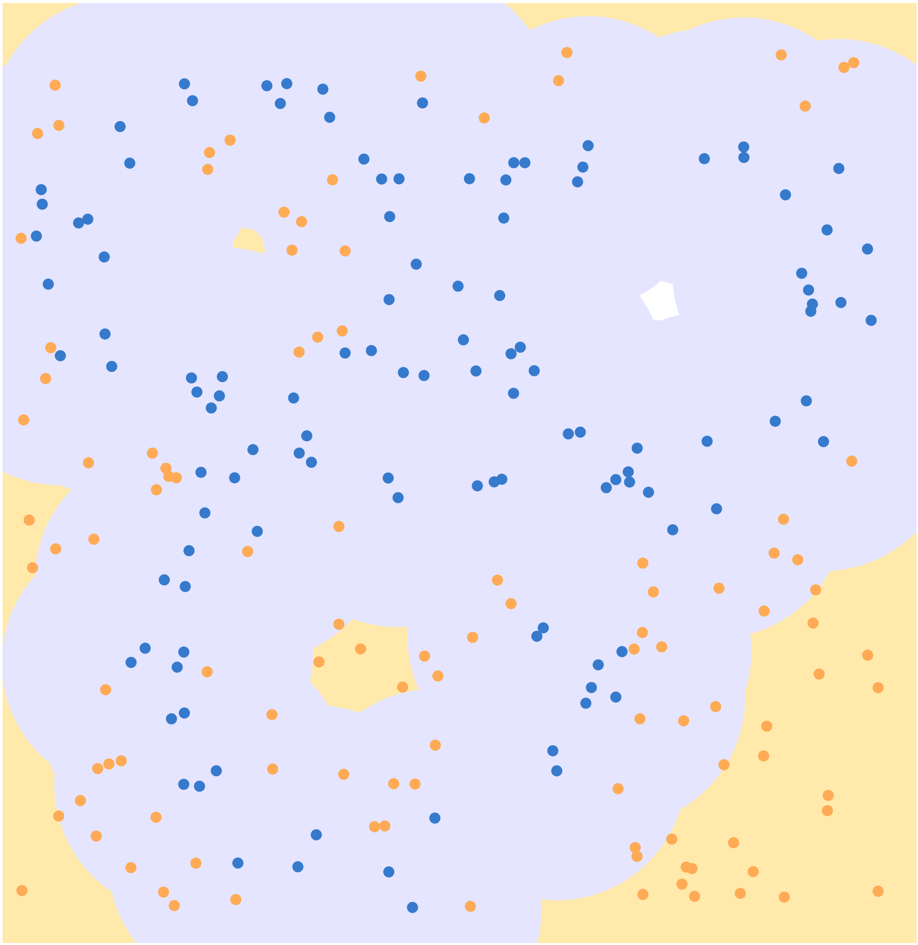

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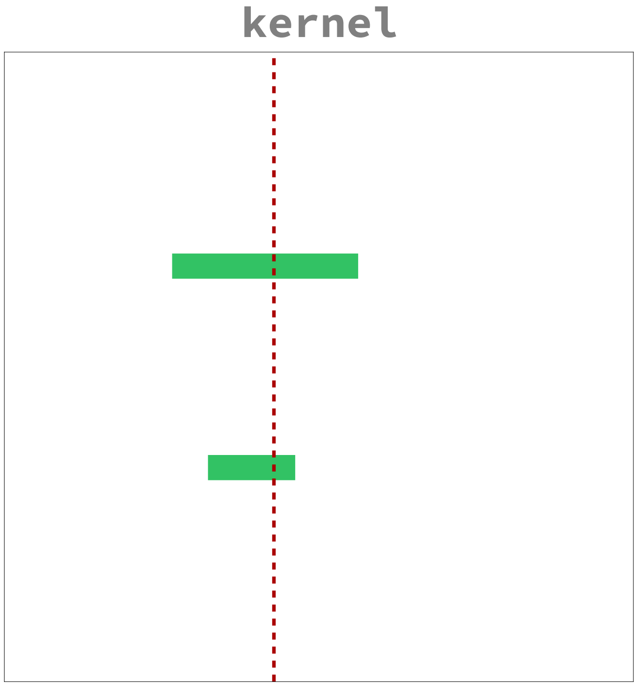

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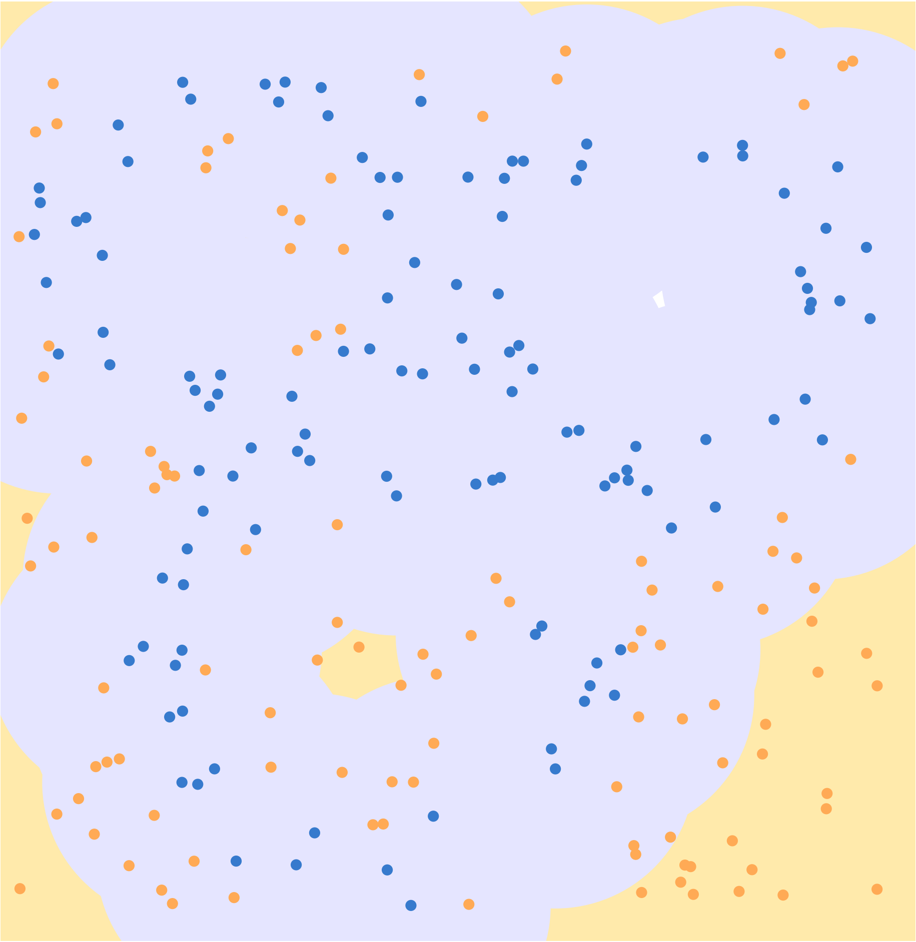

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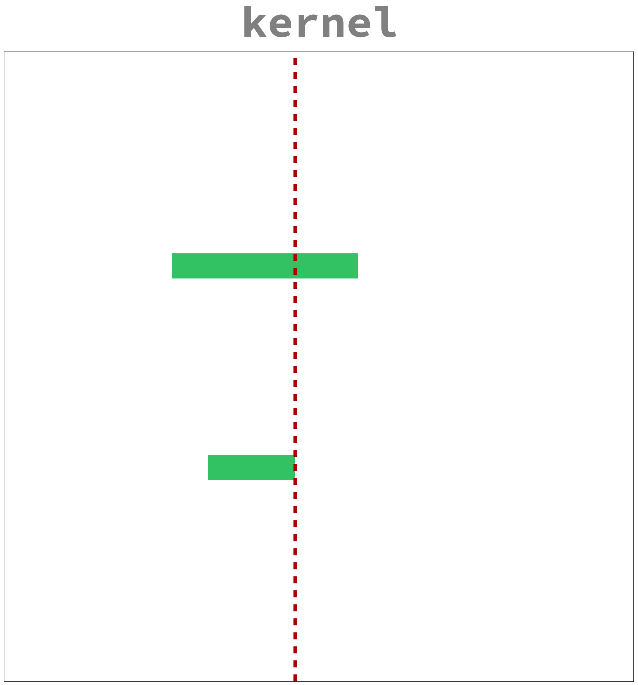

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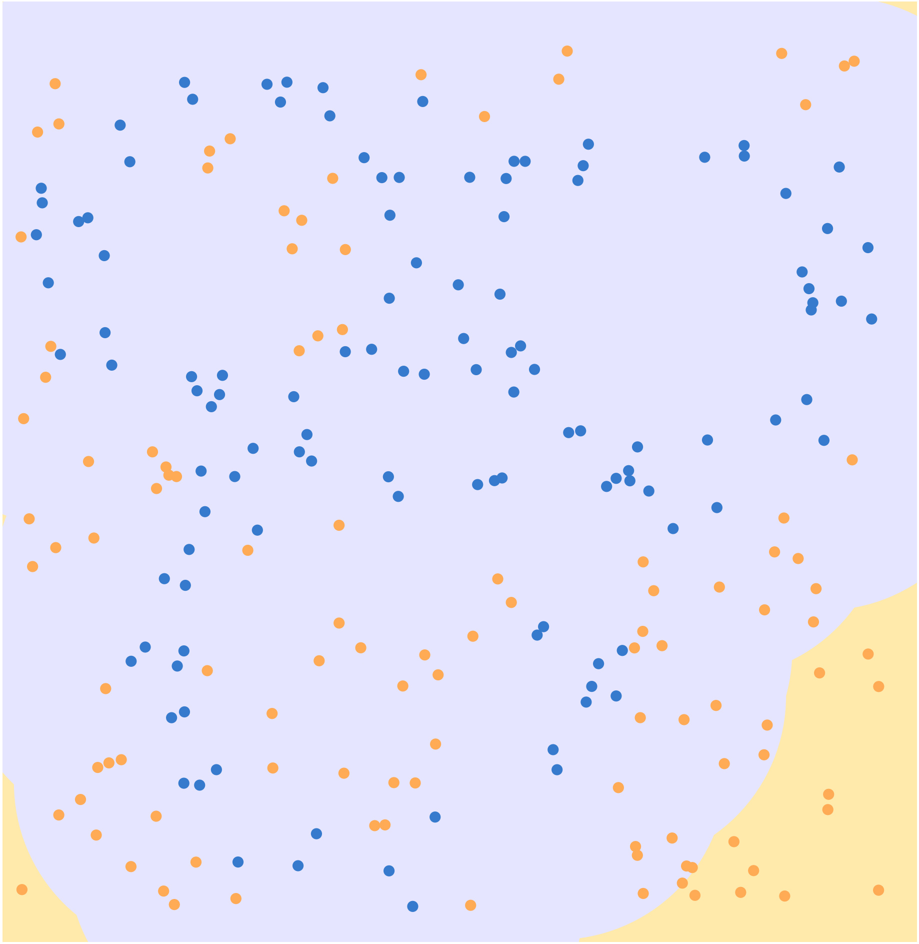

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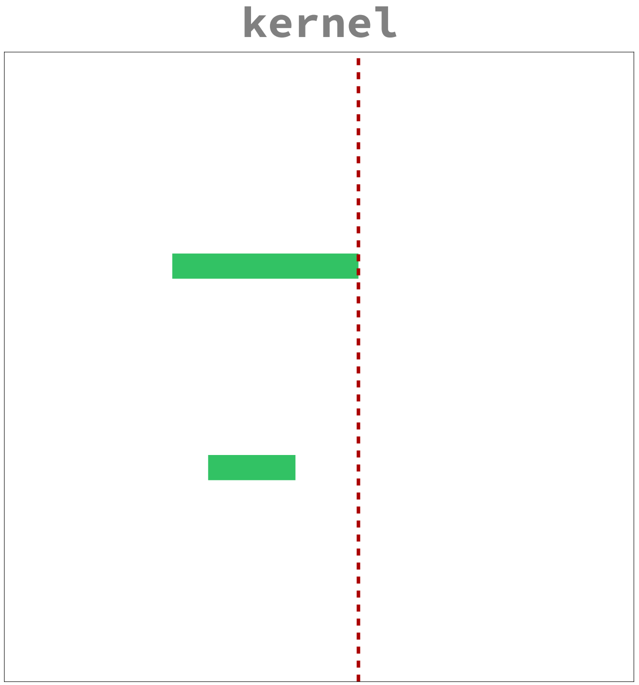

Those "kernel" features – formally

$$\begin{array}{ccccccc} \cdots & \hookrightarrow & B_r(A_0) & \hookrightarrow & B_{r'}(A_0) & \hookrightarrow & \cdots \\ & & {\scriptstyle i_r}\big\downarrow & & {\scriptstyle i_{r'}}\big\downarrow & & \\ \cdots & \hookrightarrow & B_r(A_0 \cup A_1) & \hookrightarrow & B_{r'}(A_0 \cup A_1) & \hookrightarrow & \cdots \end{array}$$

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$$\cdots \longrightarrow \mathrm{im}(i_r^*) \longrightarrow \mathrm{im}(i_{r'}^*) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{coker}(i_r^*) \longrightarrow \mathrm{coker}(i_{r'}^*) \longrightarrow \cdots$$

Computing kernel, image, cokernel PH

- Algorithm in [1] (2009)
- Setting for the algorithm
  - simplicial complex K
  - subcomplex $L \leq K$
  - filtration function f on K
  - L filtered by the restriction of f

[1] D. Cohen-Steiner, H. Edelsbrunner, J. Harer, and D. Morozov, "*Persistent homology for kernels, images, and cokernels*," in Proc. 20th Ann. ACM-SIAM Sympos. Discrete Alg., 2009, pp. 1011–1020

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$$K_r = f^{-1}[0, r], \;\; L_r = L \cap f^{-1}[0, r]$$

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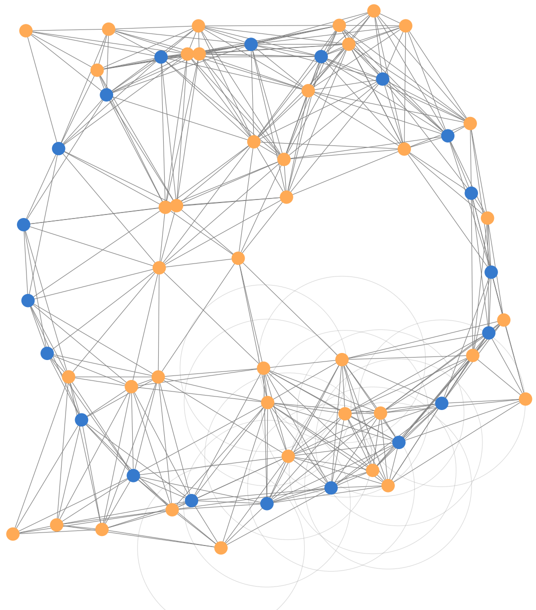
Čech complex? yes, but too big



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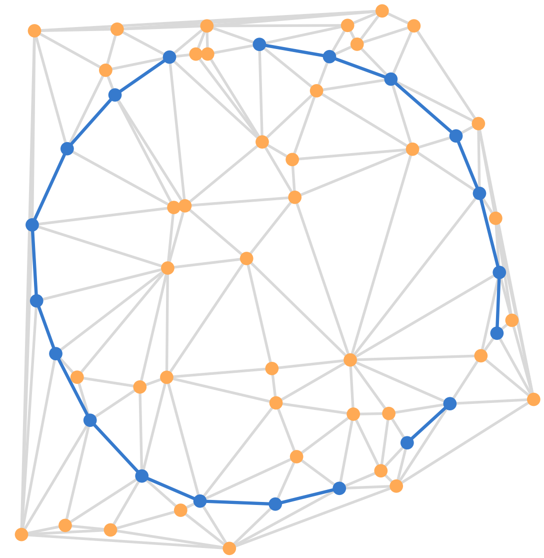
standard Delaunay/alpha complex? no



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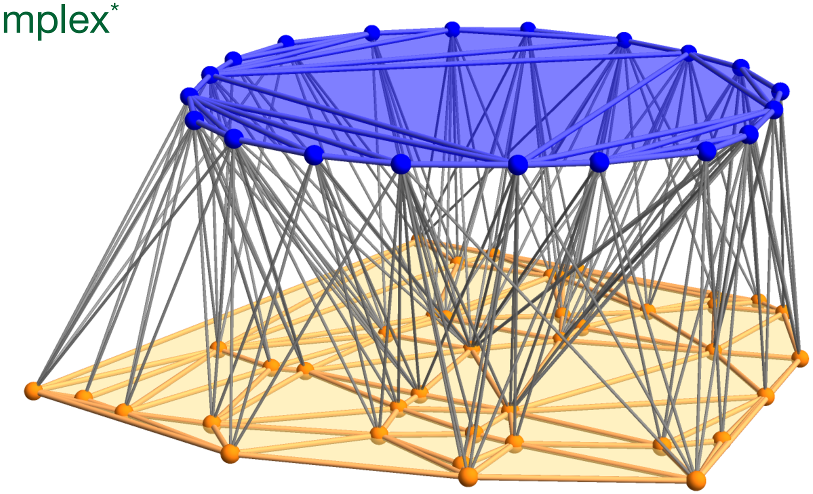
Chromatic Alpha Complex*



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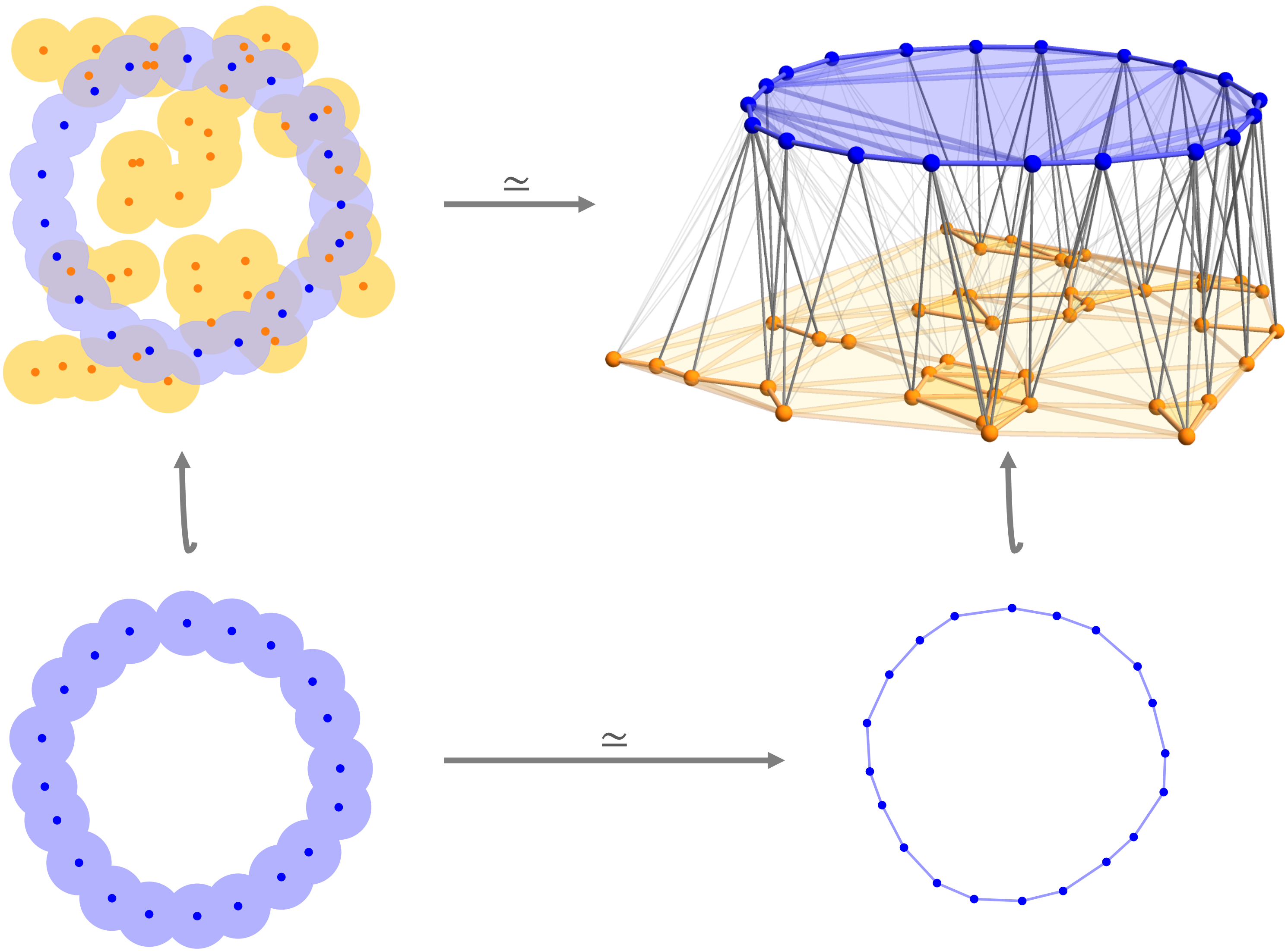
(*) For two colors defined by Y. Reani and O. Bobrowski as “A coupled alpha complex” in 2021. We generalize the construction to any number of colors.

Chromatic Alpha Complex



Chromatic Delaunay complex, $\mathrm{Del}(\chi)$

- A is a point set, σ is a set of colors, $\chi : A \to \sigma$ is a chromatic point set

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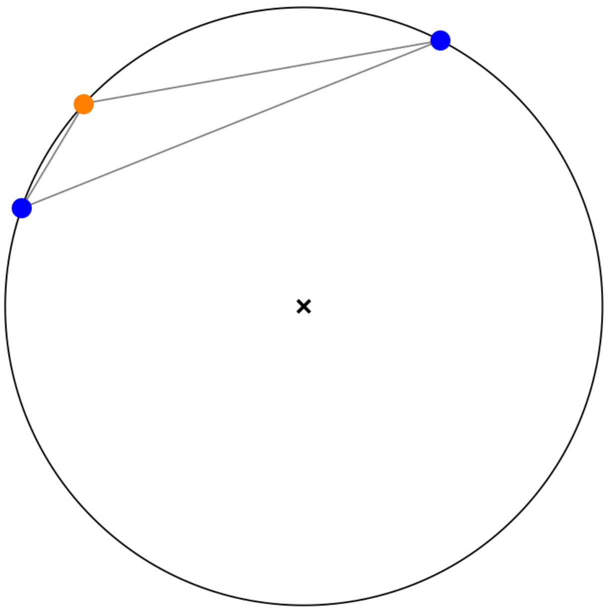
$\nu \in \mathrm{Del}(A)$

iff

exists empty sphere passing through its vertices



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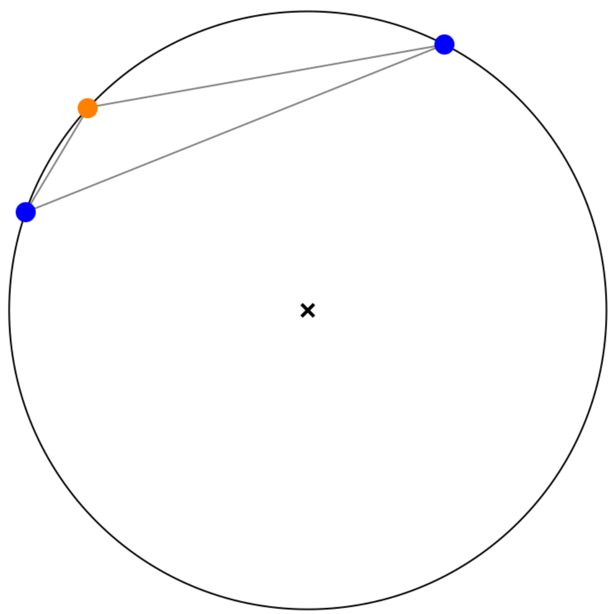

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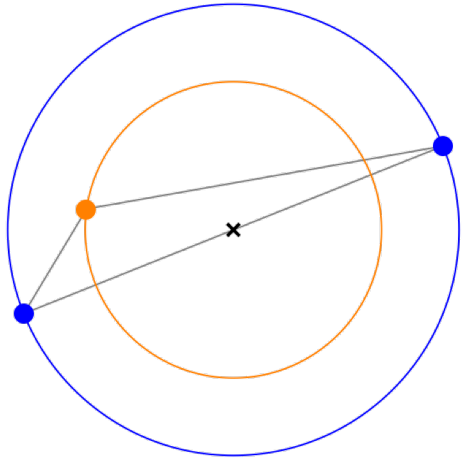
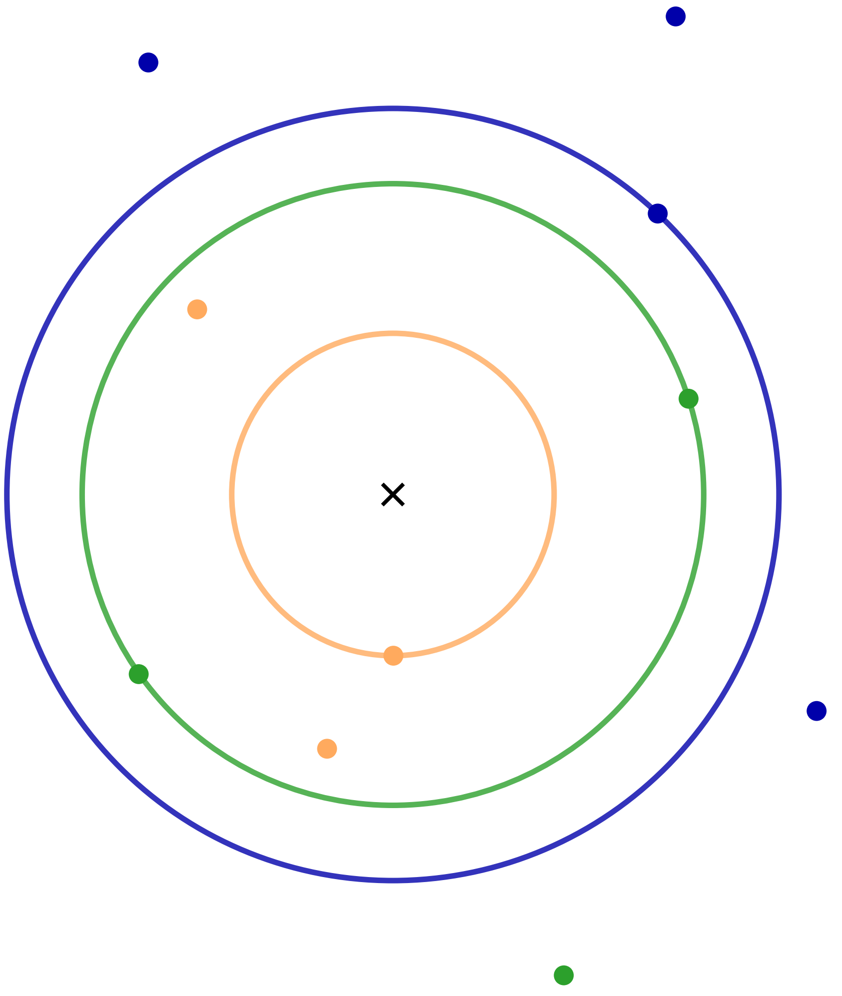
exists stack of empty spheres passing through its vertices



σ-stack of $(d-1)$-spheres in $\mathbb{R}^d$

- concentric spheres, one for each color
  - possibly with radius 0
- is *empty* if
  - each sphere empty of points **of its color**
- *passes through* points ν if
  - the s-colored points of ν lie on the s-colored sphere for each color $s \in \sigma$



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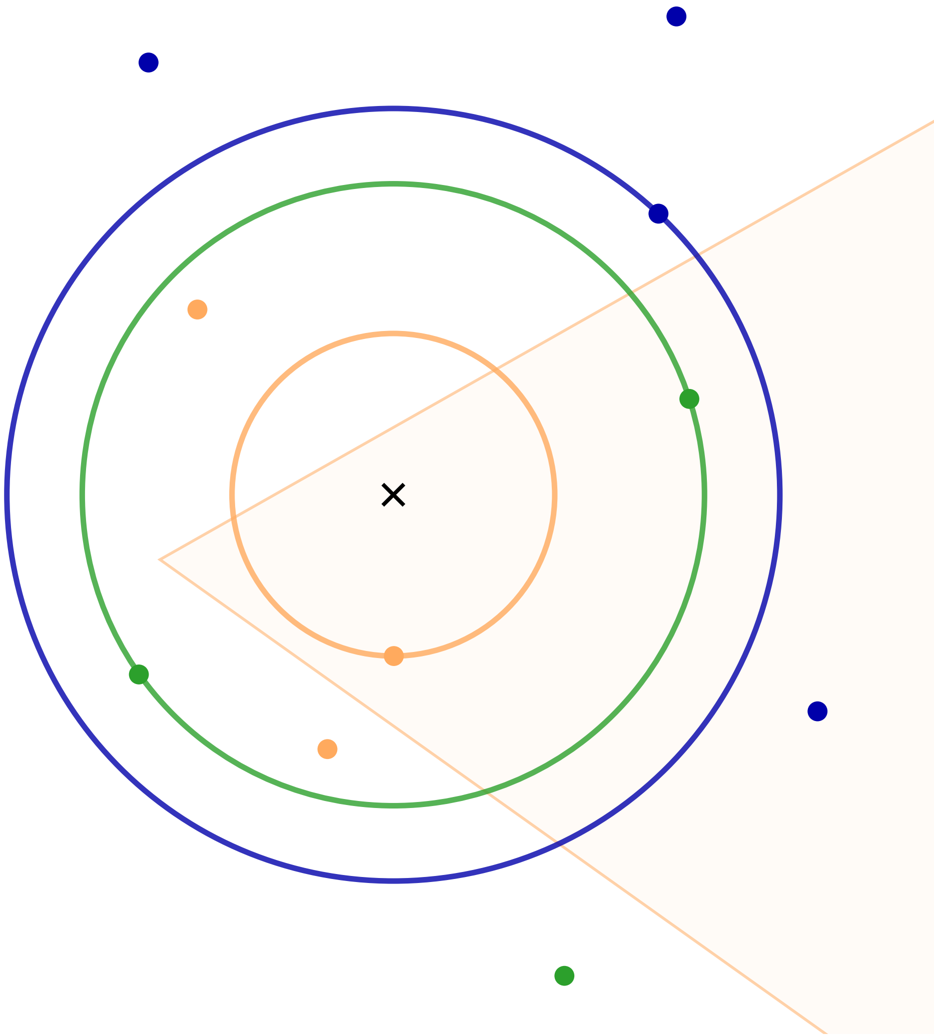

centers of empty stacks passing through ν

=

intersection of Voronoi cells of all colors

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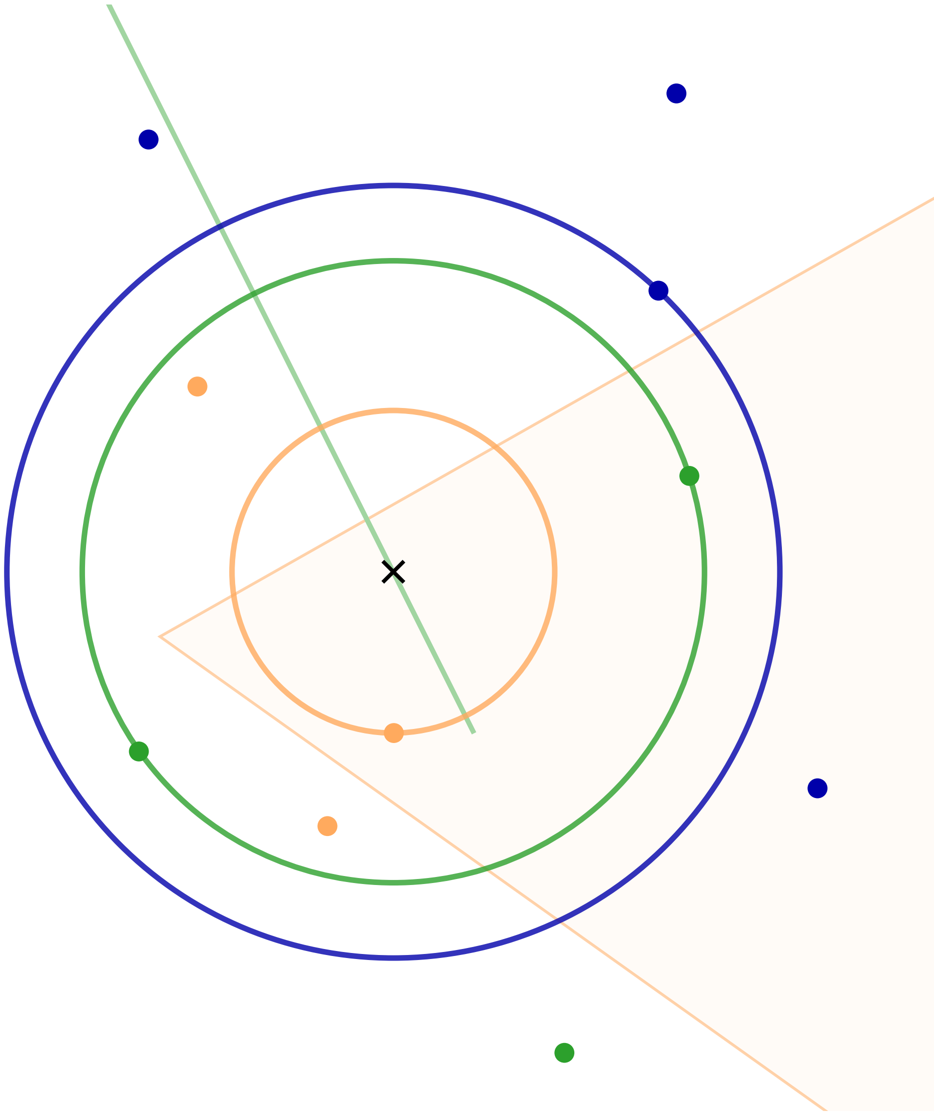

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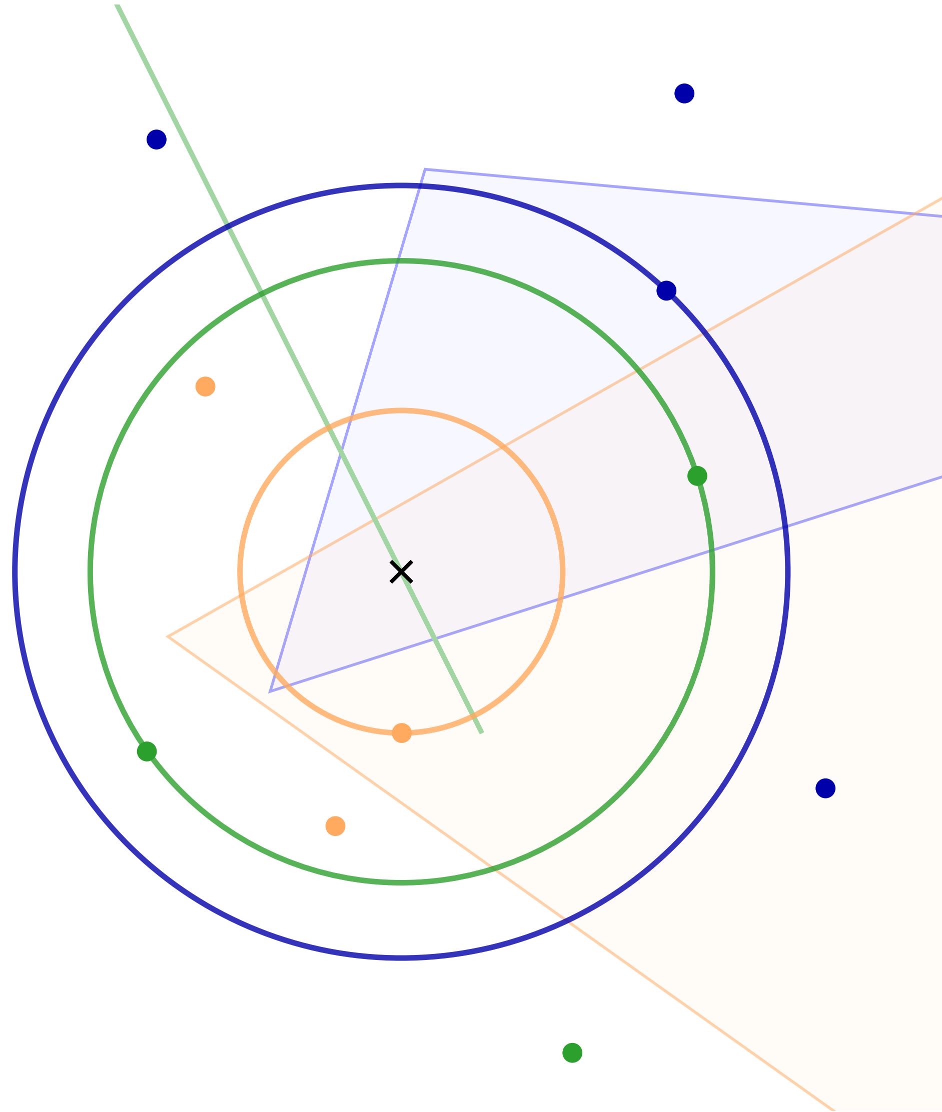

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Stack radius

- stack radius = radius of its largest sphere
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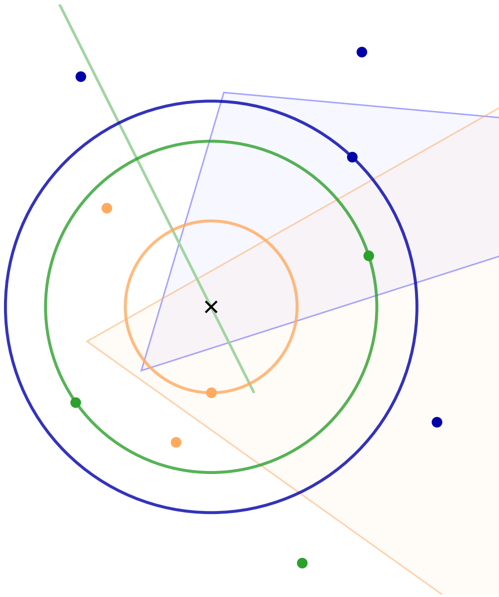

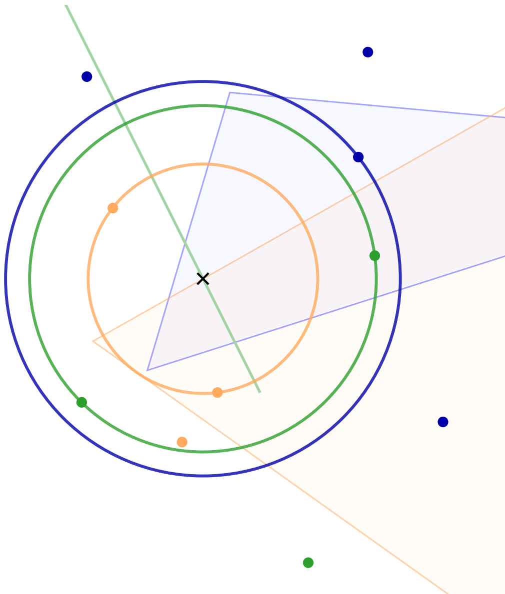

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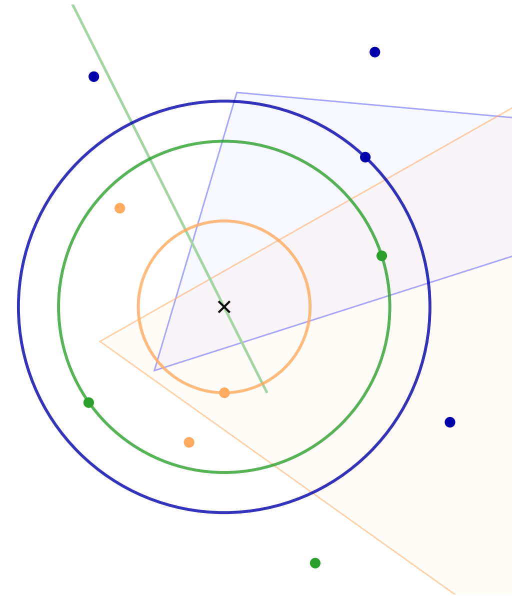
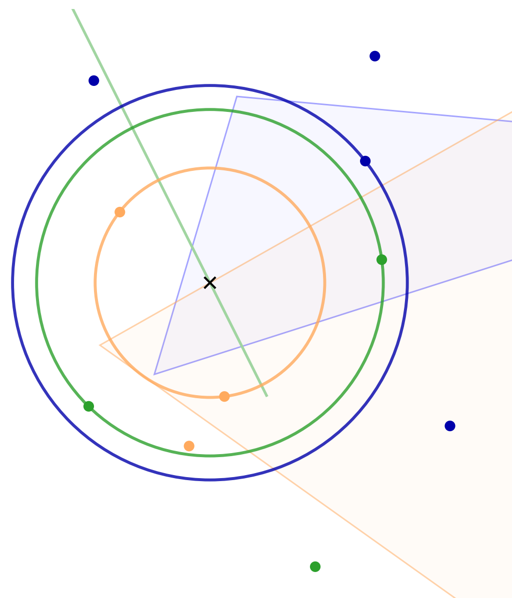
$$\mathrm{Rad} : \mathrm{Del}(\chi) \to \mathbb{R}$$

Radius function assigns to every ν the smallest radius of an empty stack passing through it





Chromatic alpha complex

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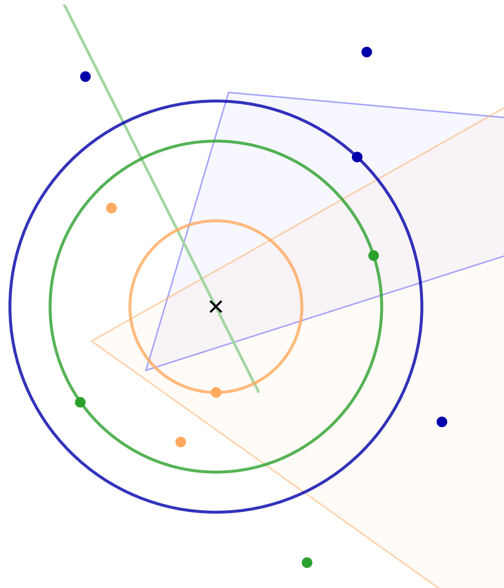

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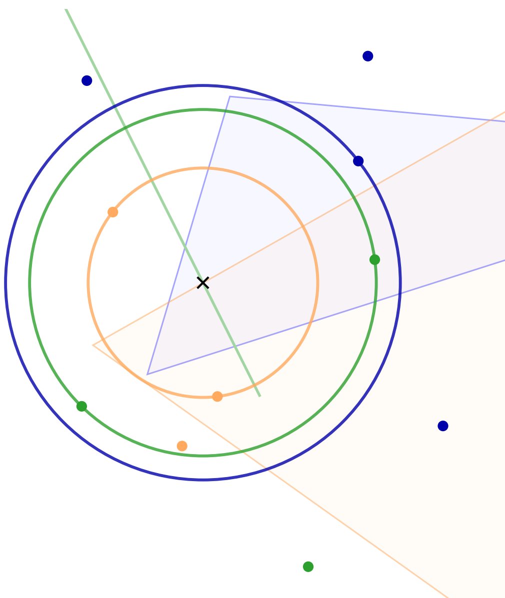

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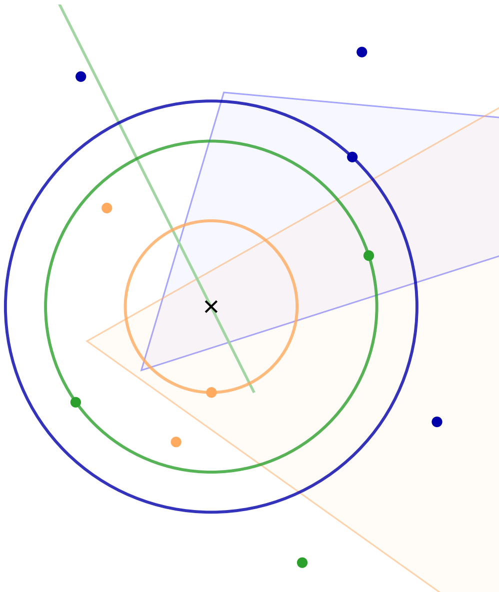
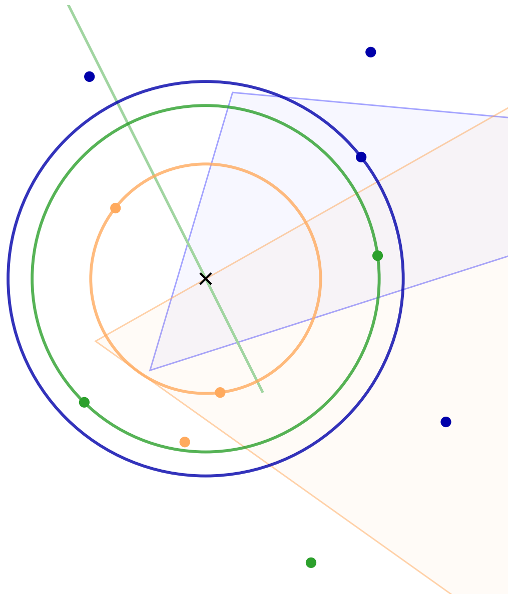
$$\mathrm{Alf}_r(A) \subseteq \mathrm{Alf}_r(\chi)$$
$$\mathrm{Alf}_r(A_i) \subseteq \mathrm{Alf}_r(\chi),$$





Chromatic alpha cplx $\simeq$ union of balls

- definition of chromatic alpha complex as the nerve of Voronoi balls

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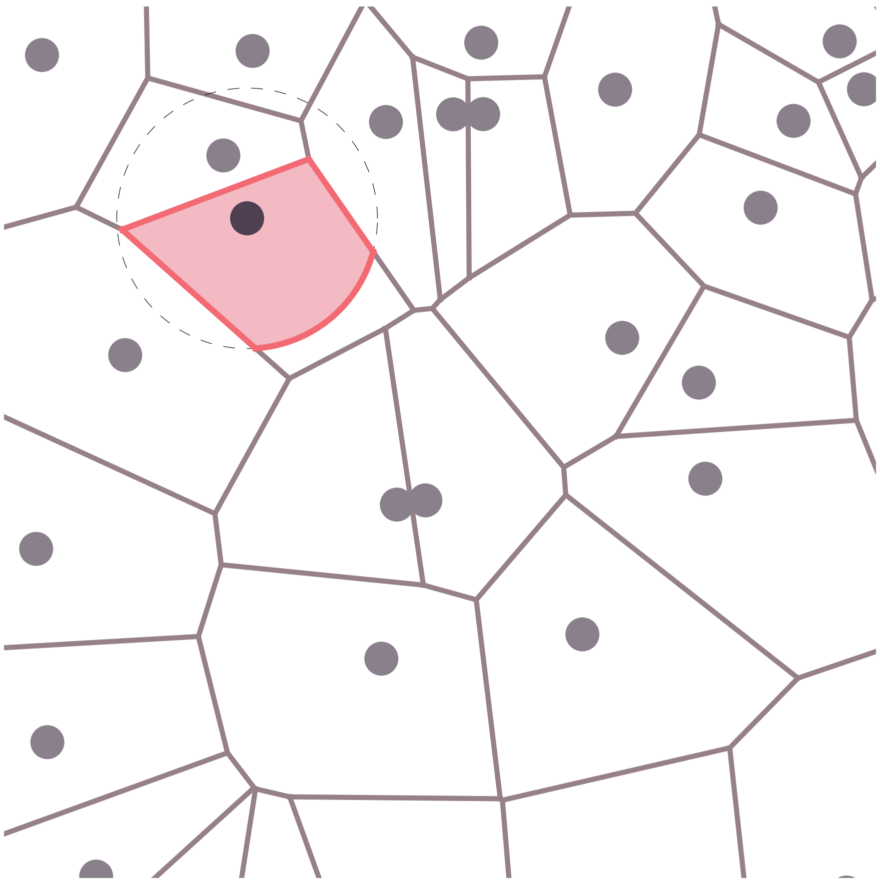

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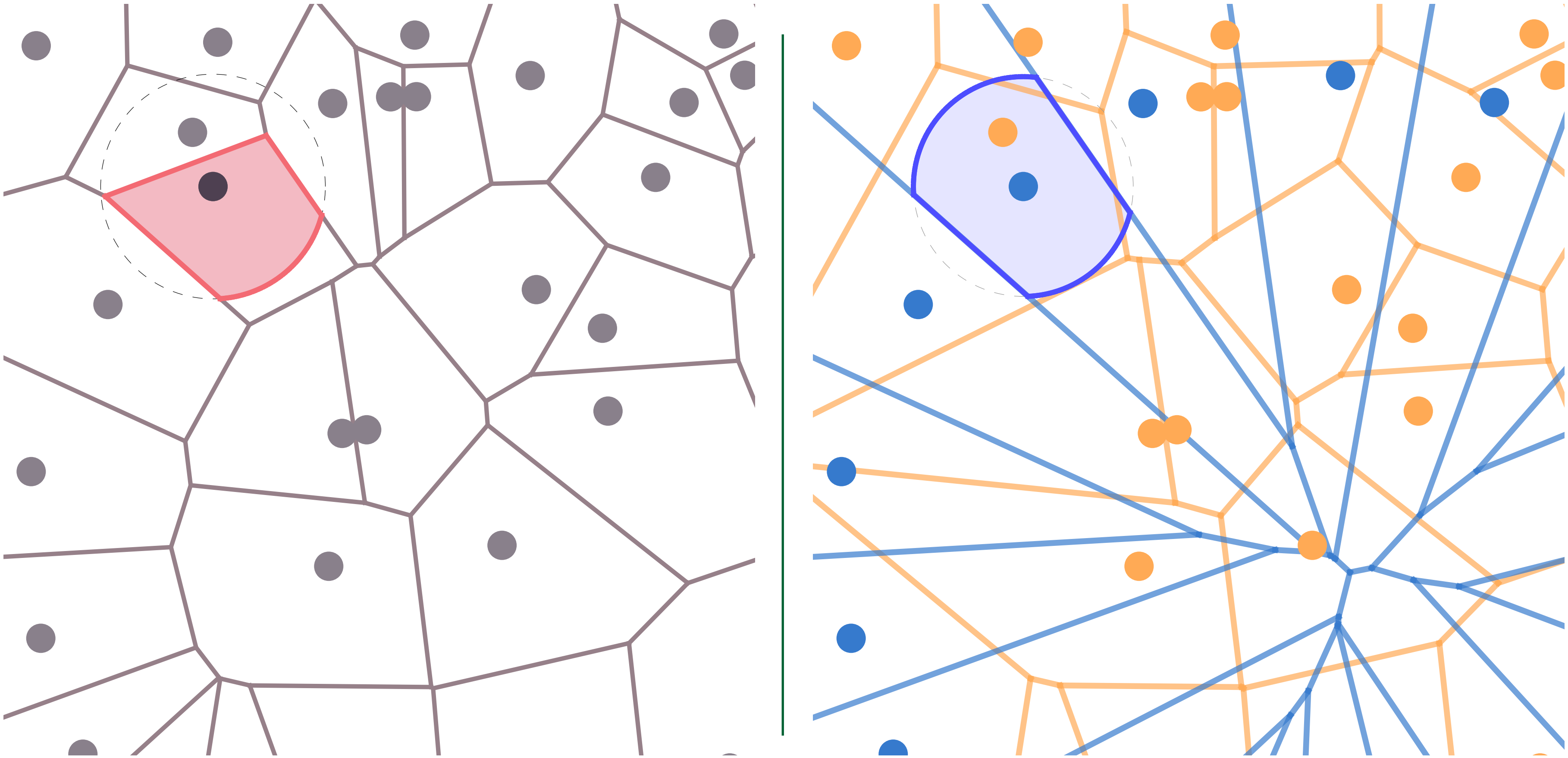
- in chromatic case we clip by Vor domain of a w.r.t. its color $\chi(a)$



Chromatic alpha cplx ≃ union of balls

Voronoi balls of $\nu \subseteq A$ intersect

iff

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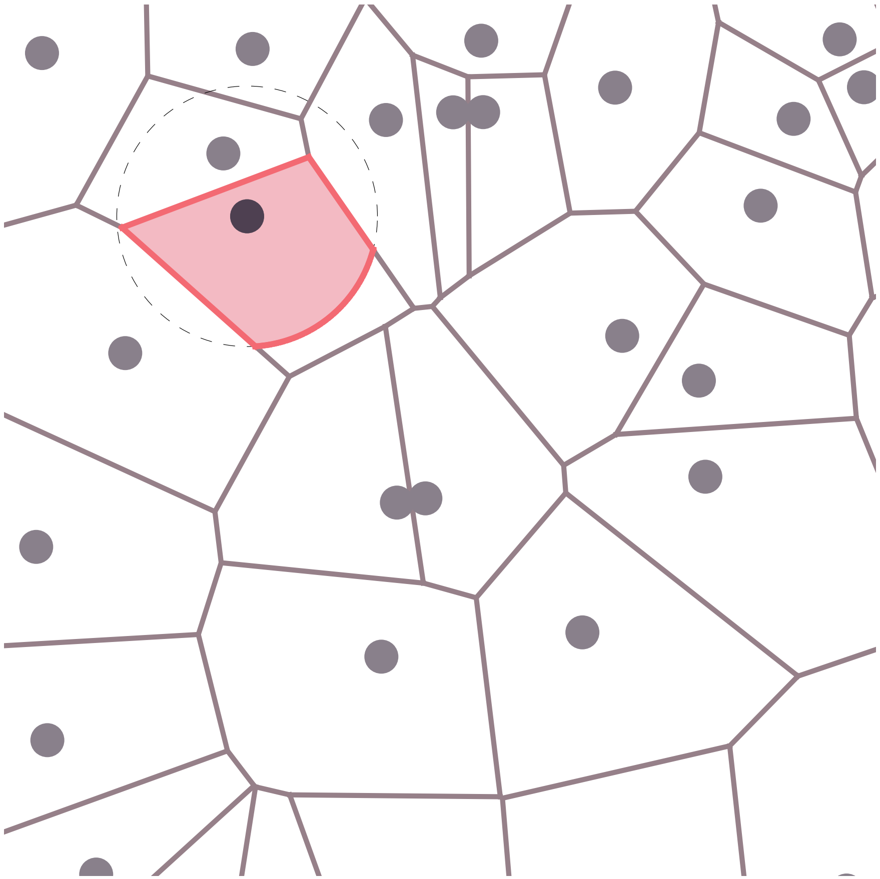
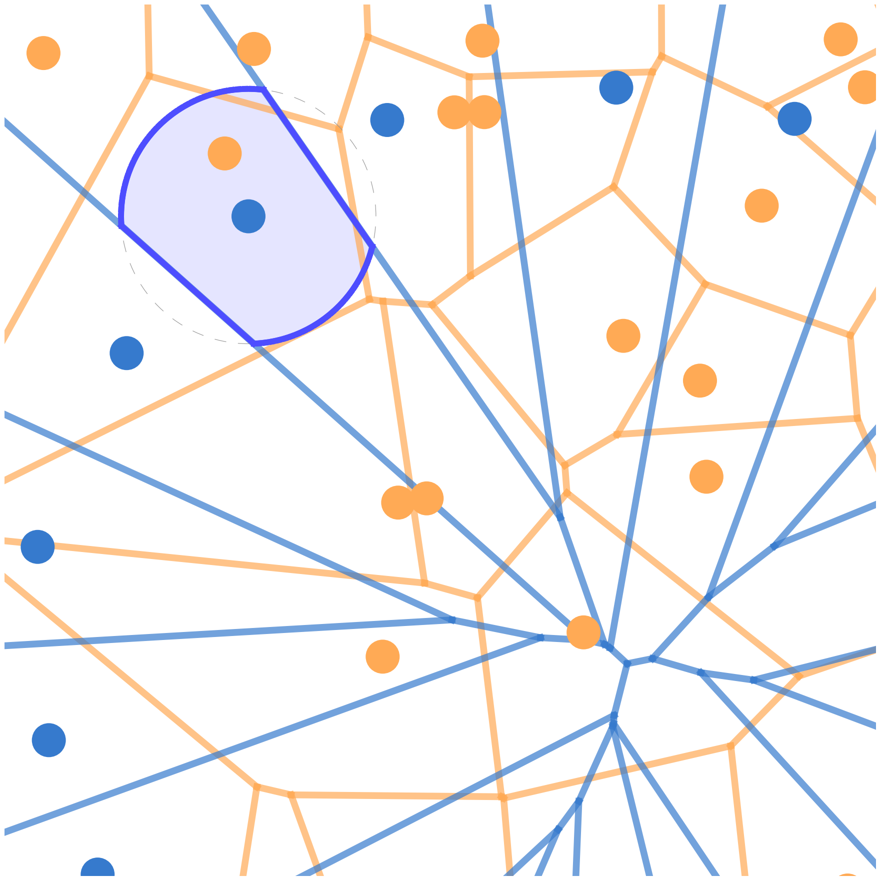
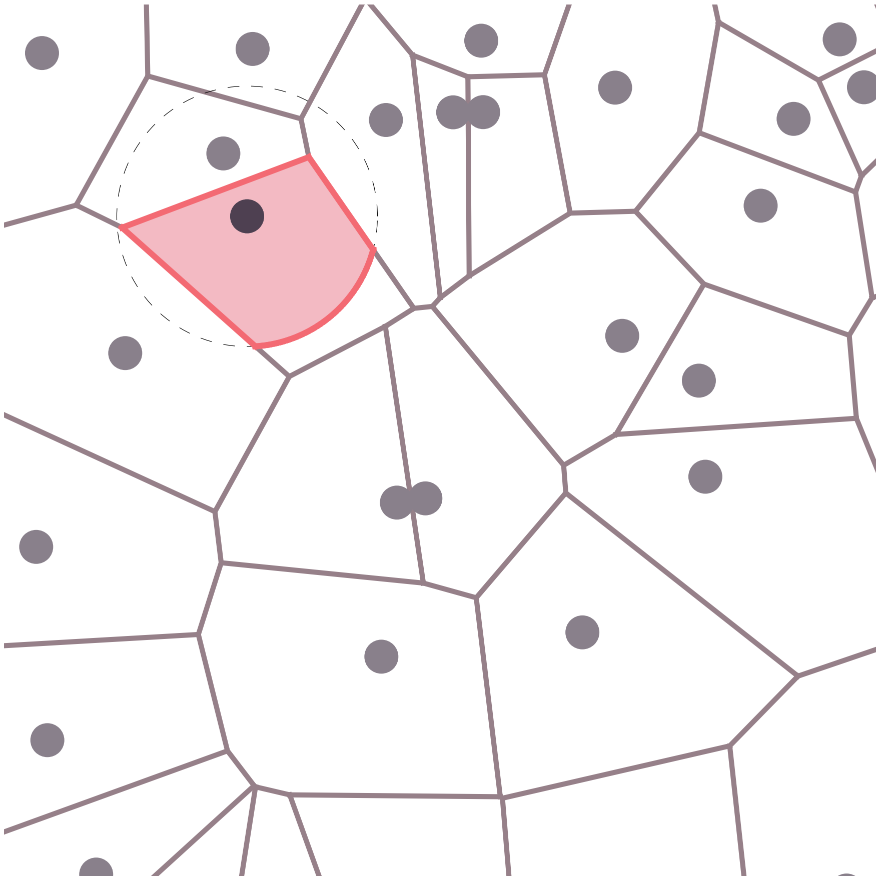
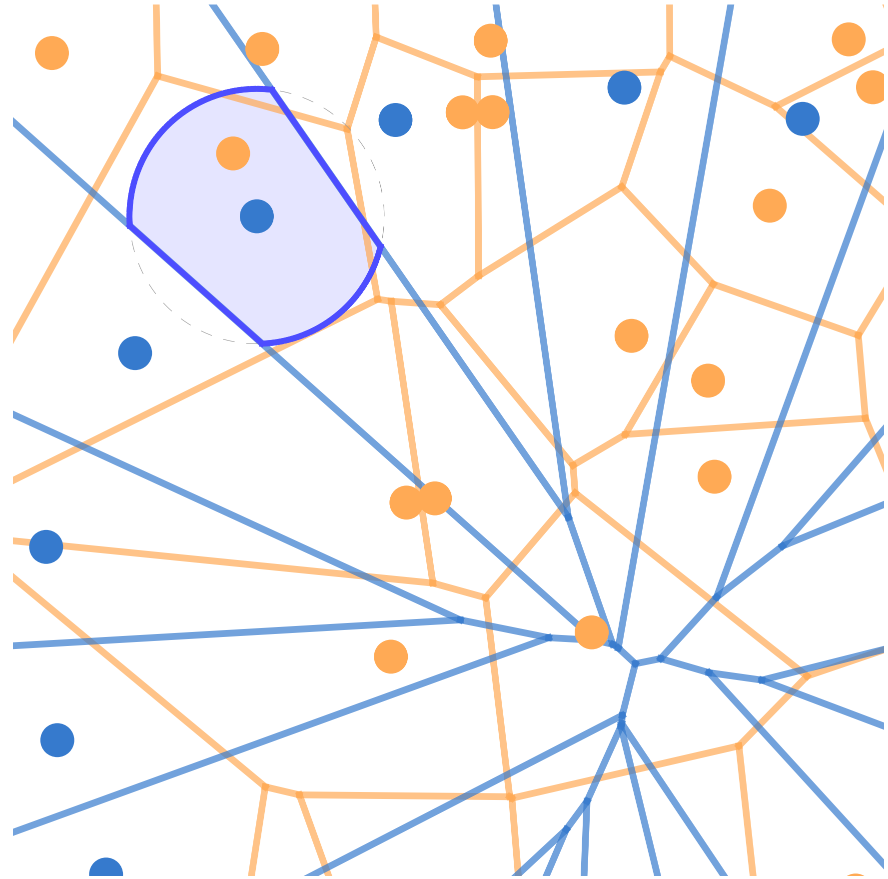




Chromatic alpha cplx ≃ union of balls

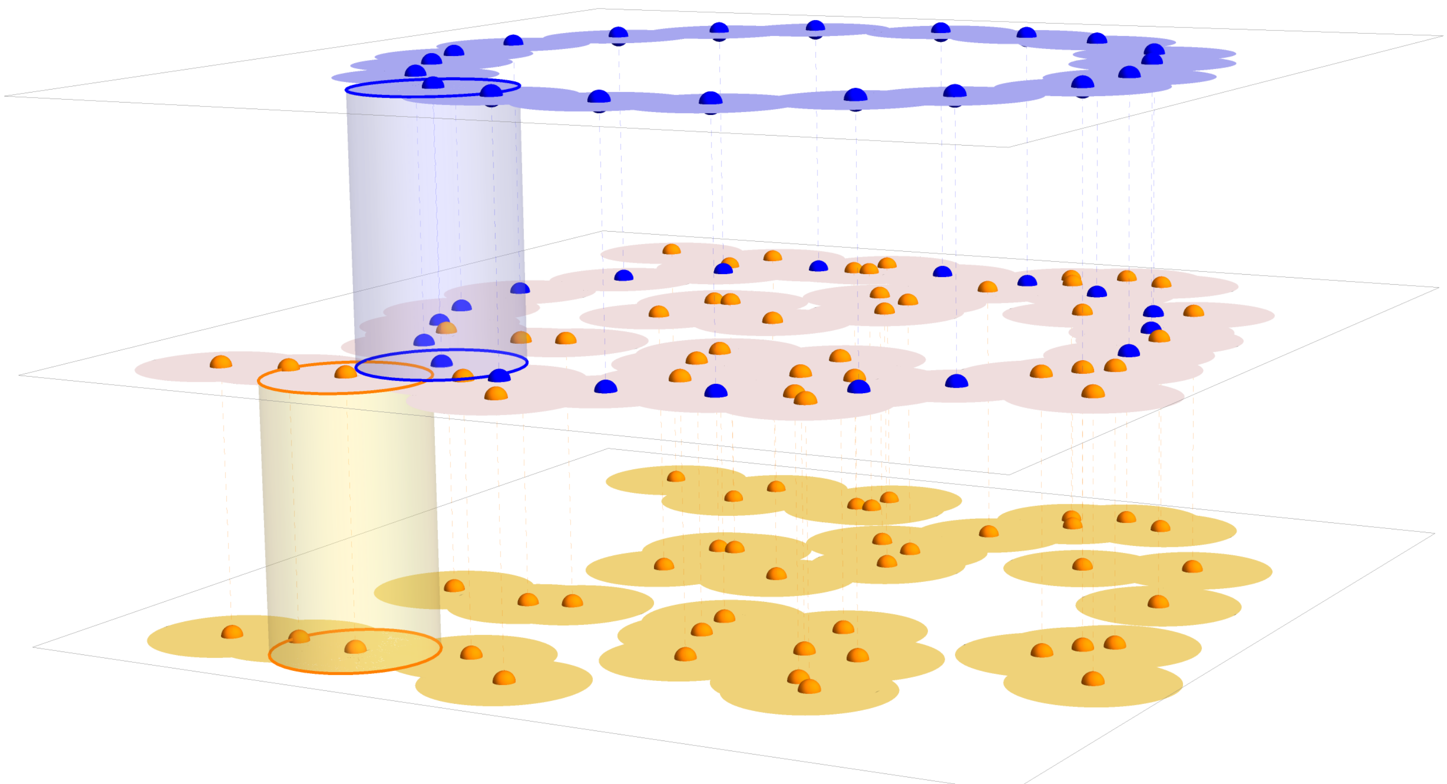
Voronoi balls of $\nu \subseteq A$ intersect iff ∃ empty sphere passing through ν	chromatic Voronoi balls of $\nu \subseteq A$ intersect iff ∃ empty stack passing through ν





Chromatic alpha cplx $\simeq$ union of balls

- $\mathrm{Alf}_r(\chi)$ is the nerve of chromatic Voronoi balls of radius r
- the union of chromatic Voronoi balls = the union of balls
- Nerve Theorem $\Rightarrow$ $\boxed{\mathrm{Alf}_r(\chi) \simeq \bigcup_{a \in A} B_r(a)}$



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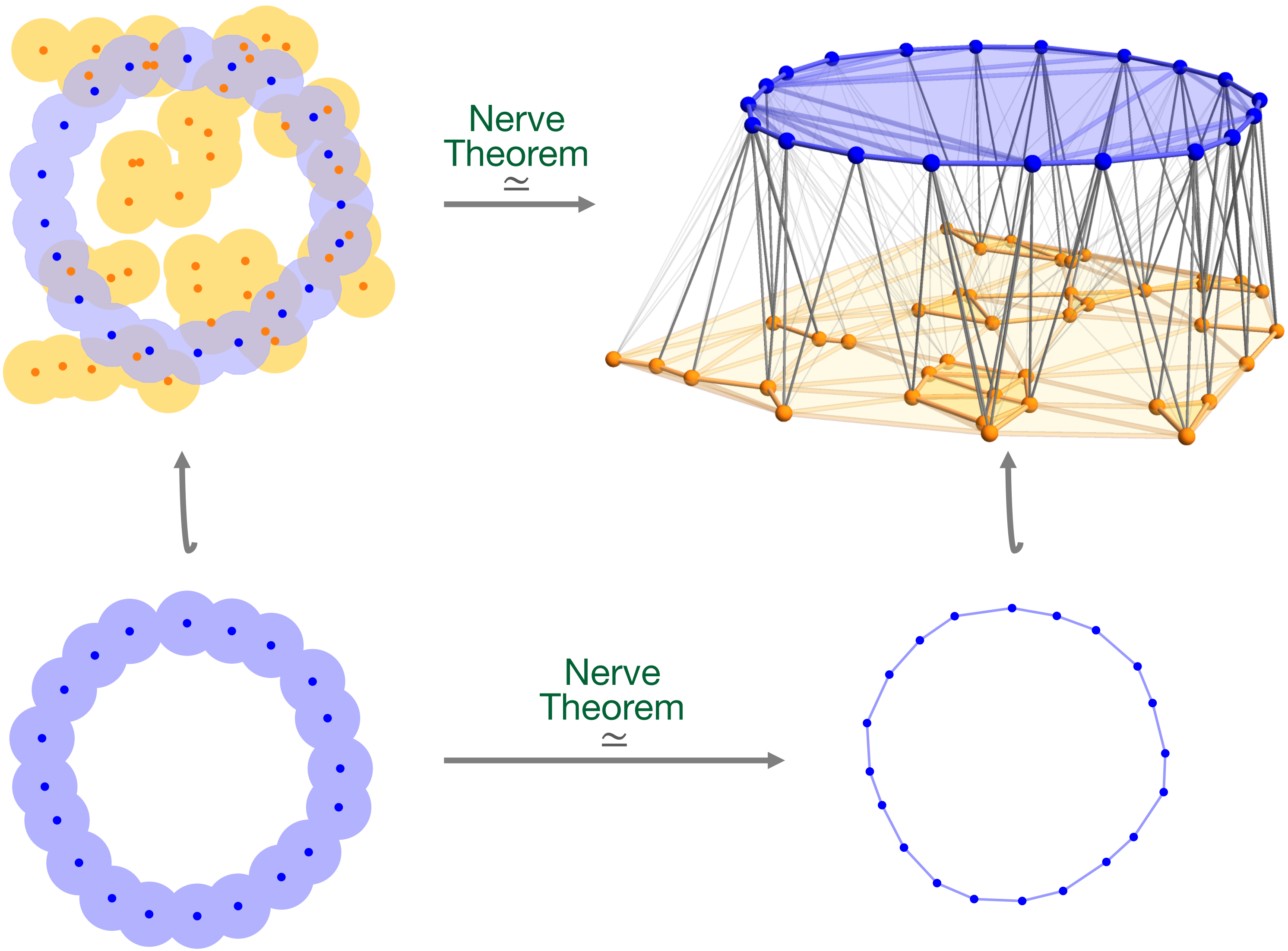

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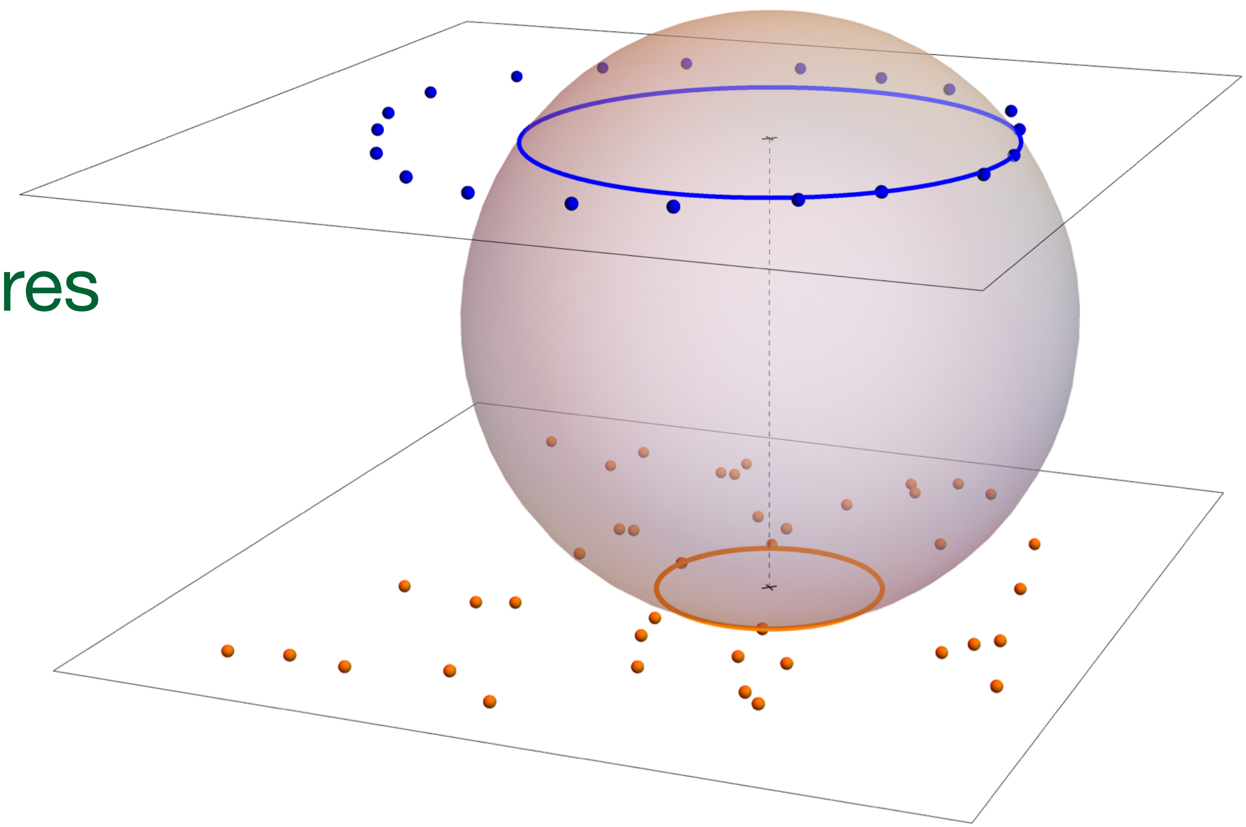

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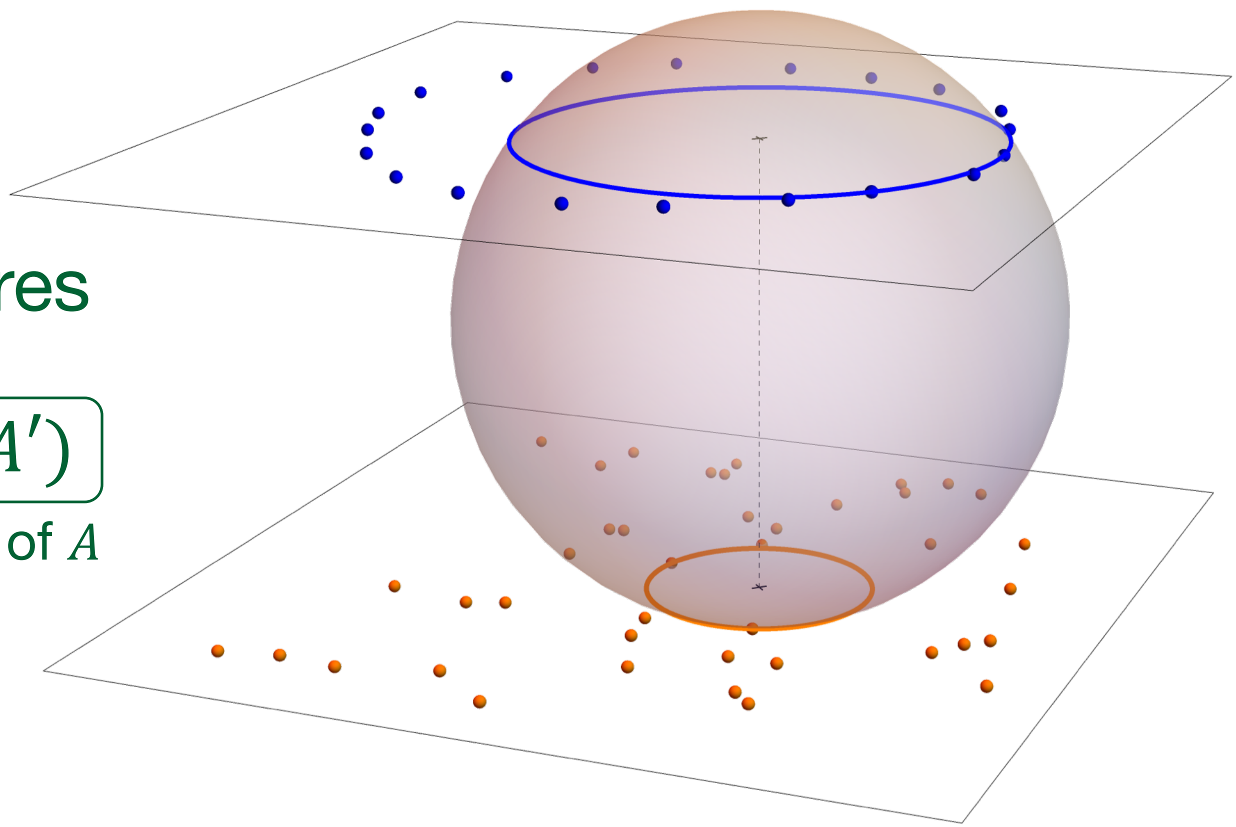
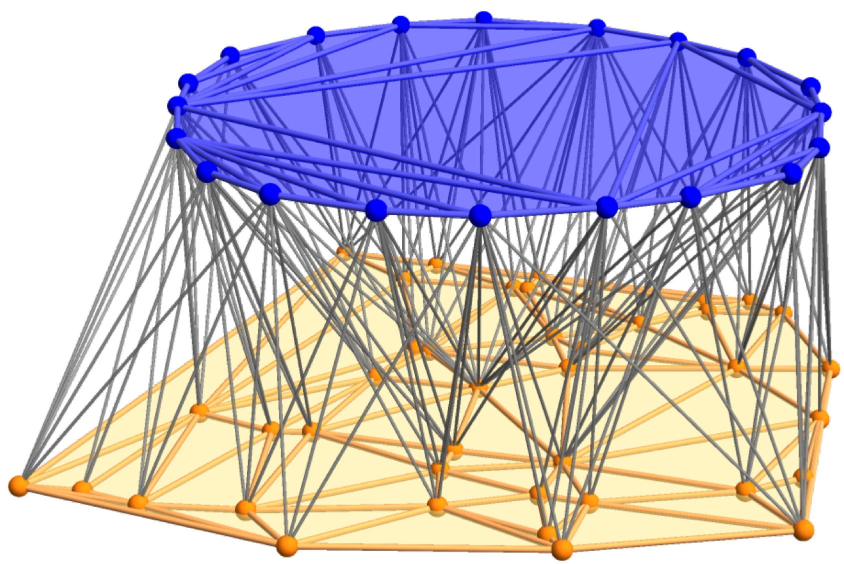
$$\mathrm{Del}(\chi) = \mathrm{Del}(A')$$

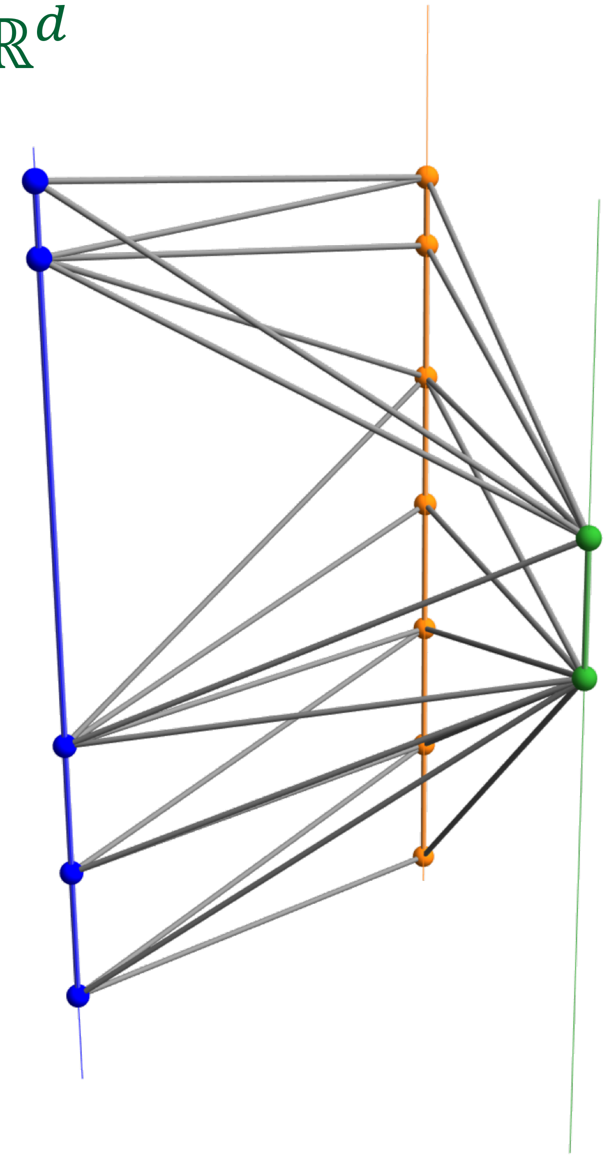
where A' is the lifting of A



Lifting in general

- Chromatic set $\chi : A \to \sigma,\ \ A \subseteq \mathbb{R}^d$
- Lift $A' \subseteq \mathbb{R}^{d+\#\sigma-1}$





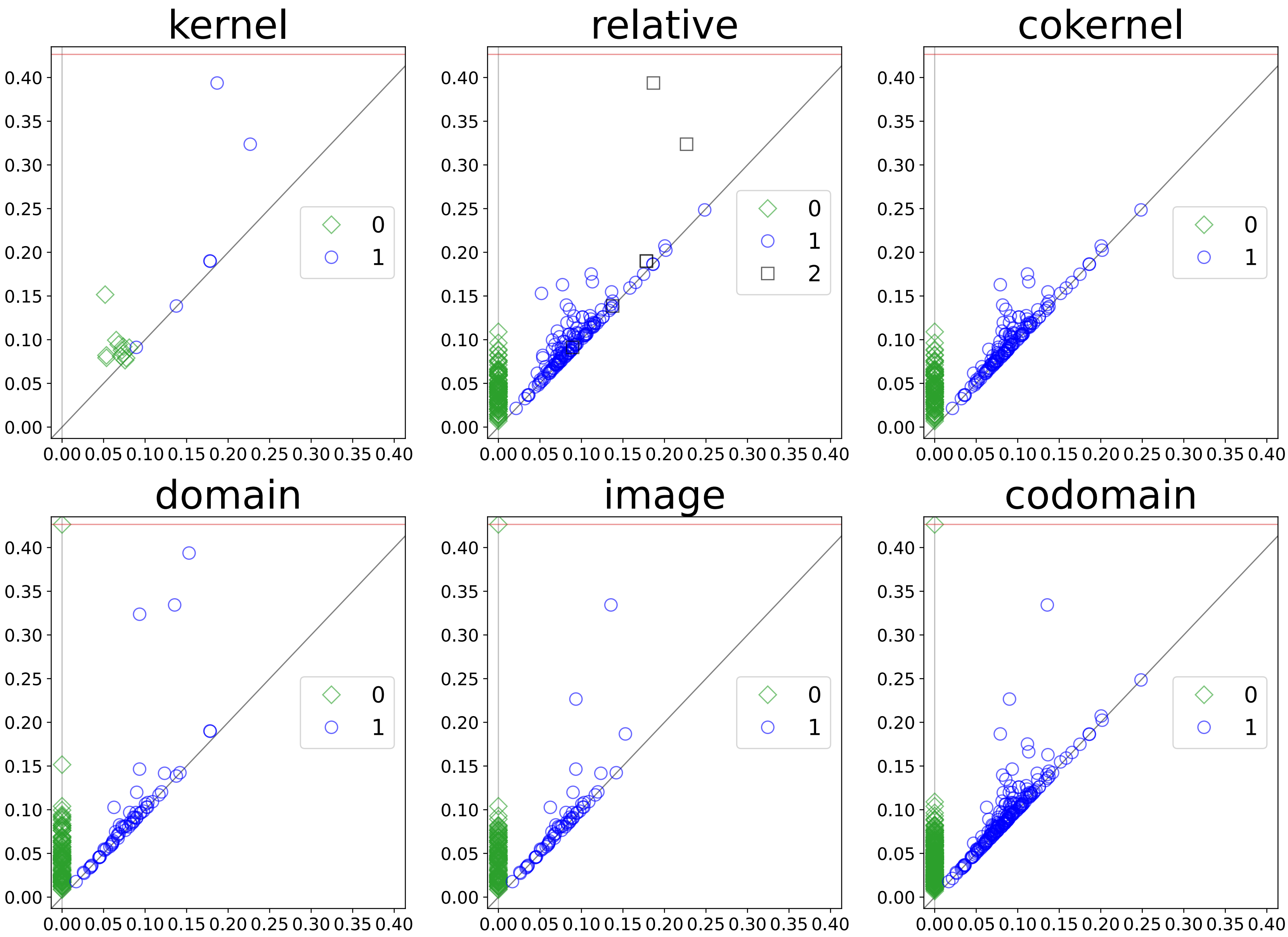
The radius function is GDMF

- K simplicial complex, $f : K \to \mathbb{R}$ monotonic
- $[\alpha, \gamma]$ is an *interval of* f if $\forall \beta \in [\alpha, \beta]$: $f(\beta) = f(\alpha)$
- f is *generalized discrete Morse function*

  if the maximal intervals of f partition K

The radius function $\mathrm{Rad} : \mathrm{Del}(\chi) \to \mathbb{R}$ is a generalized discrete Morse function.

The six-pack of persistent diagrams



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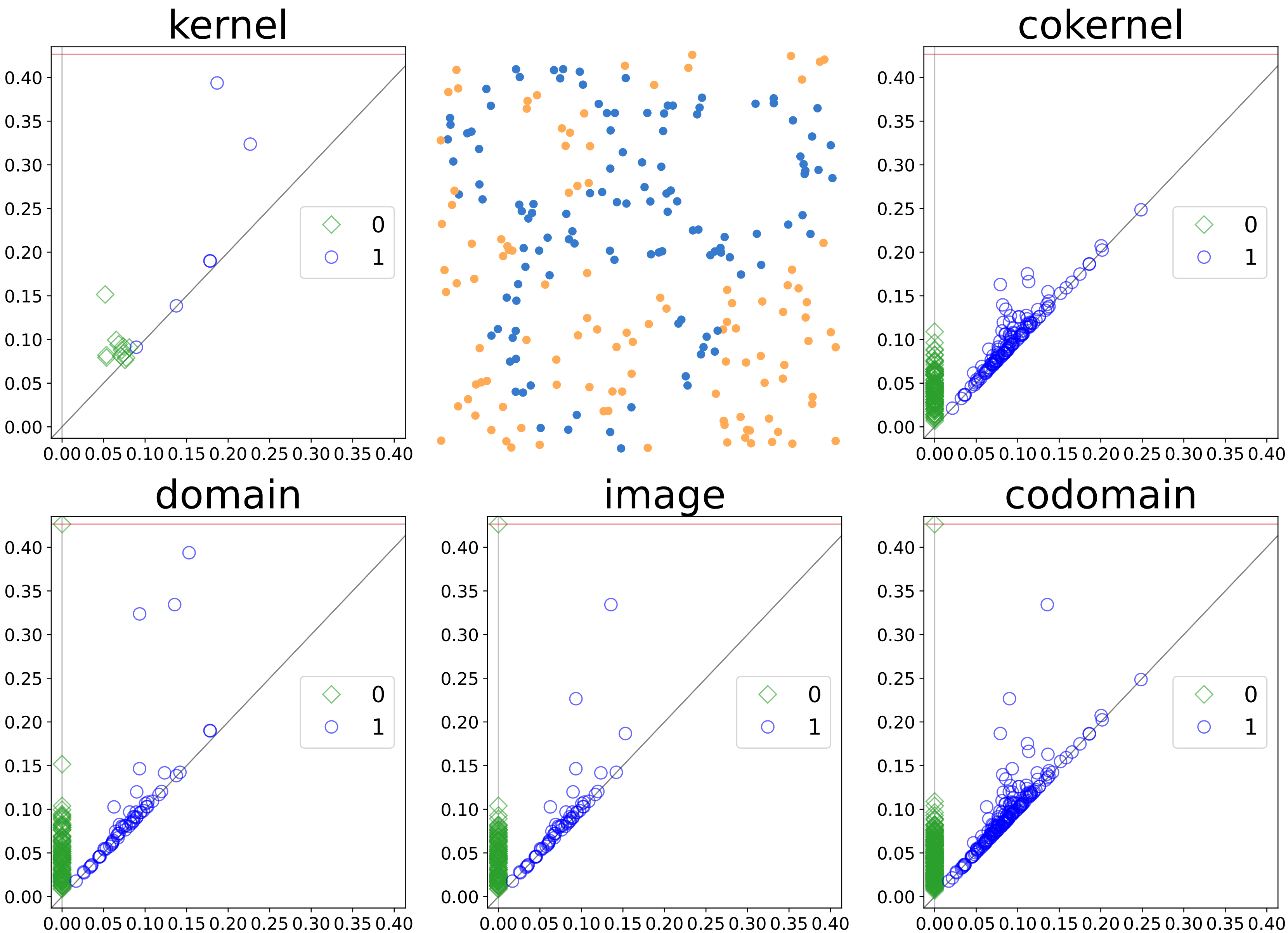

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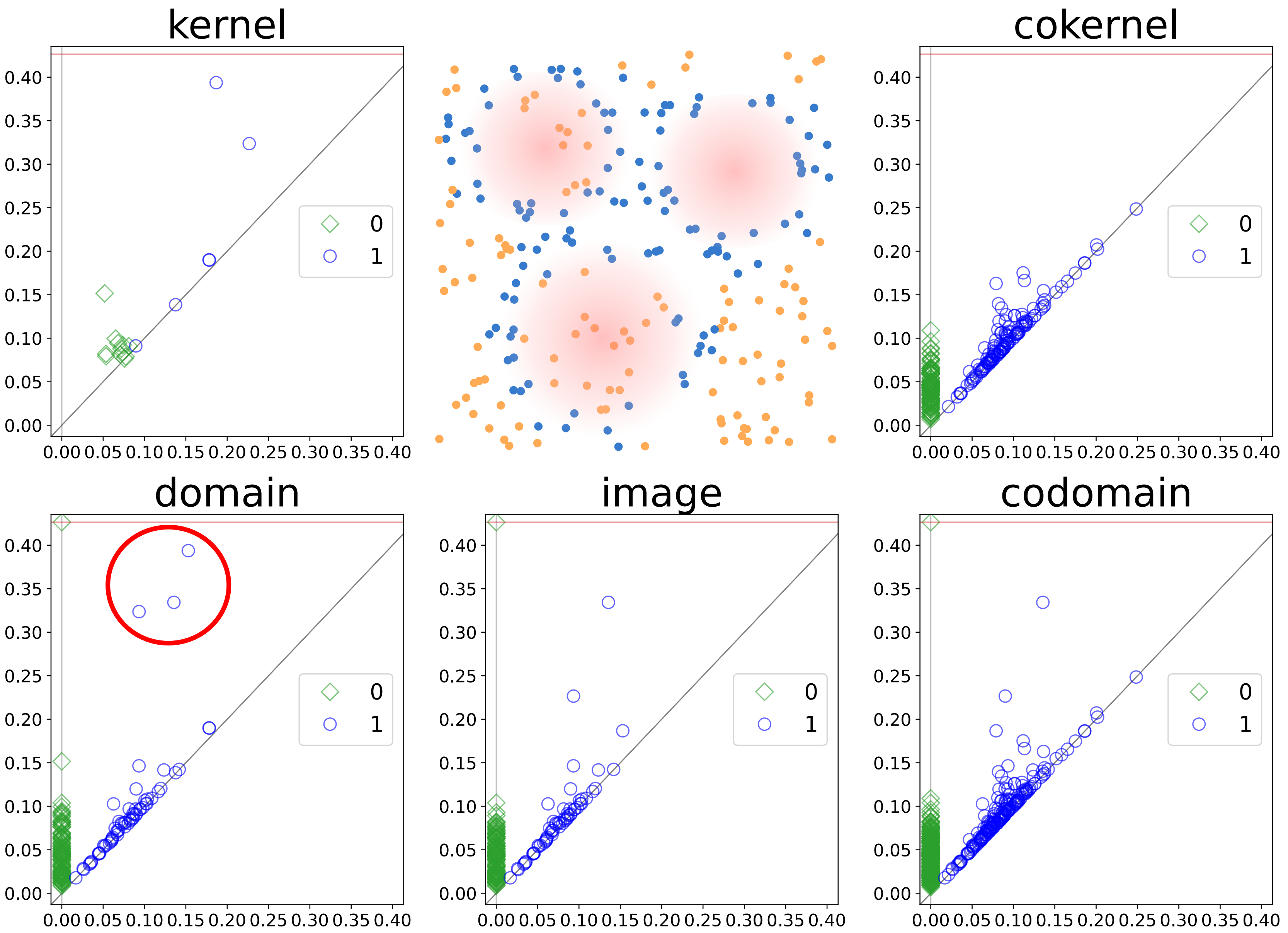

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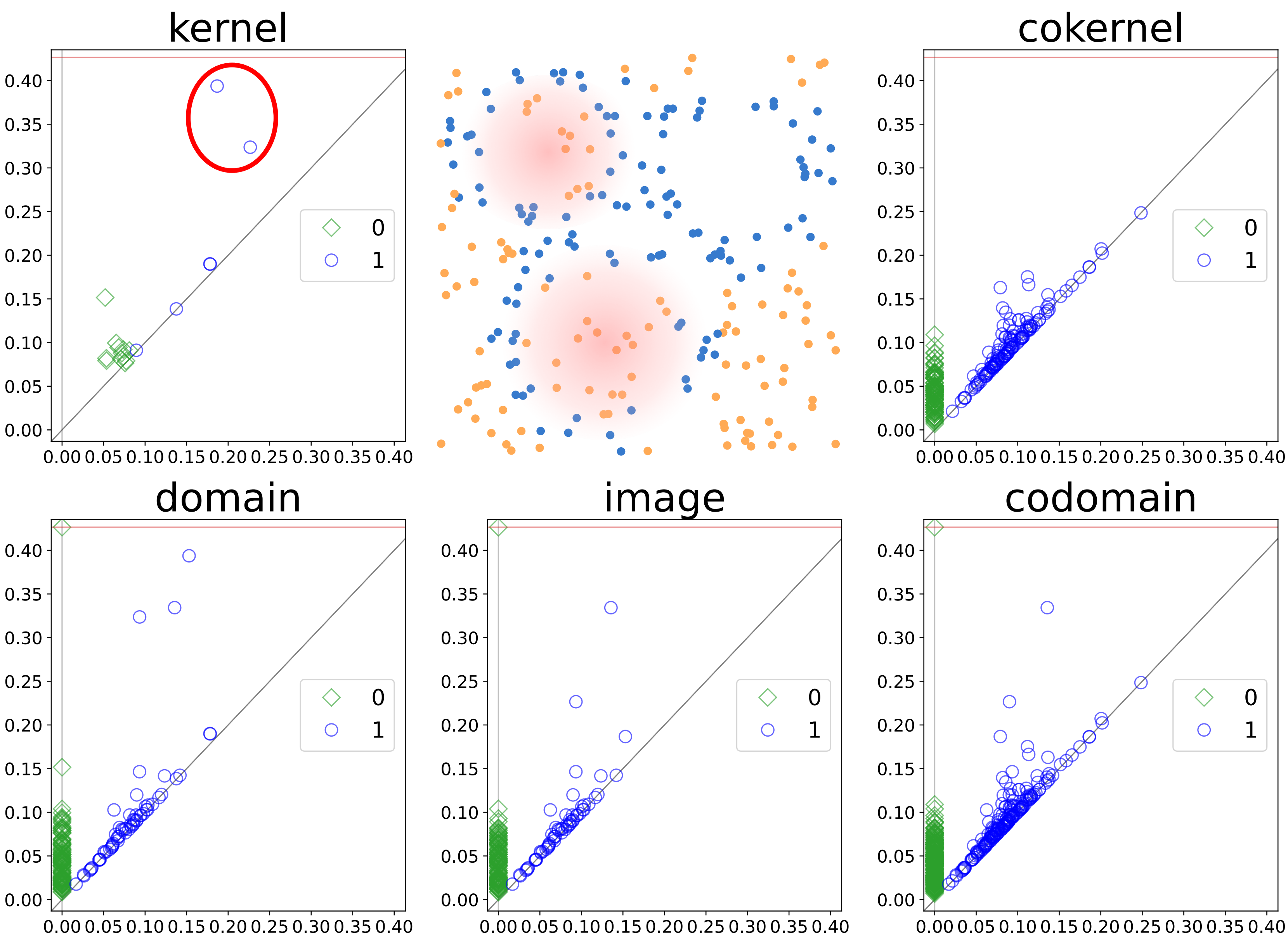

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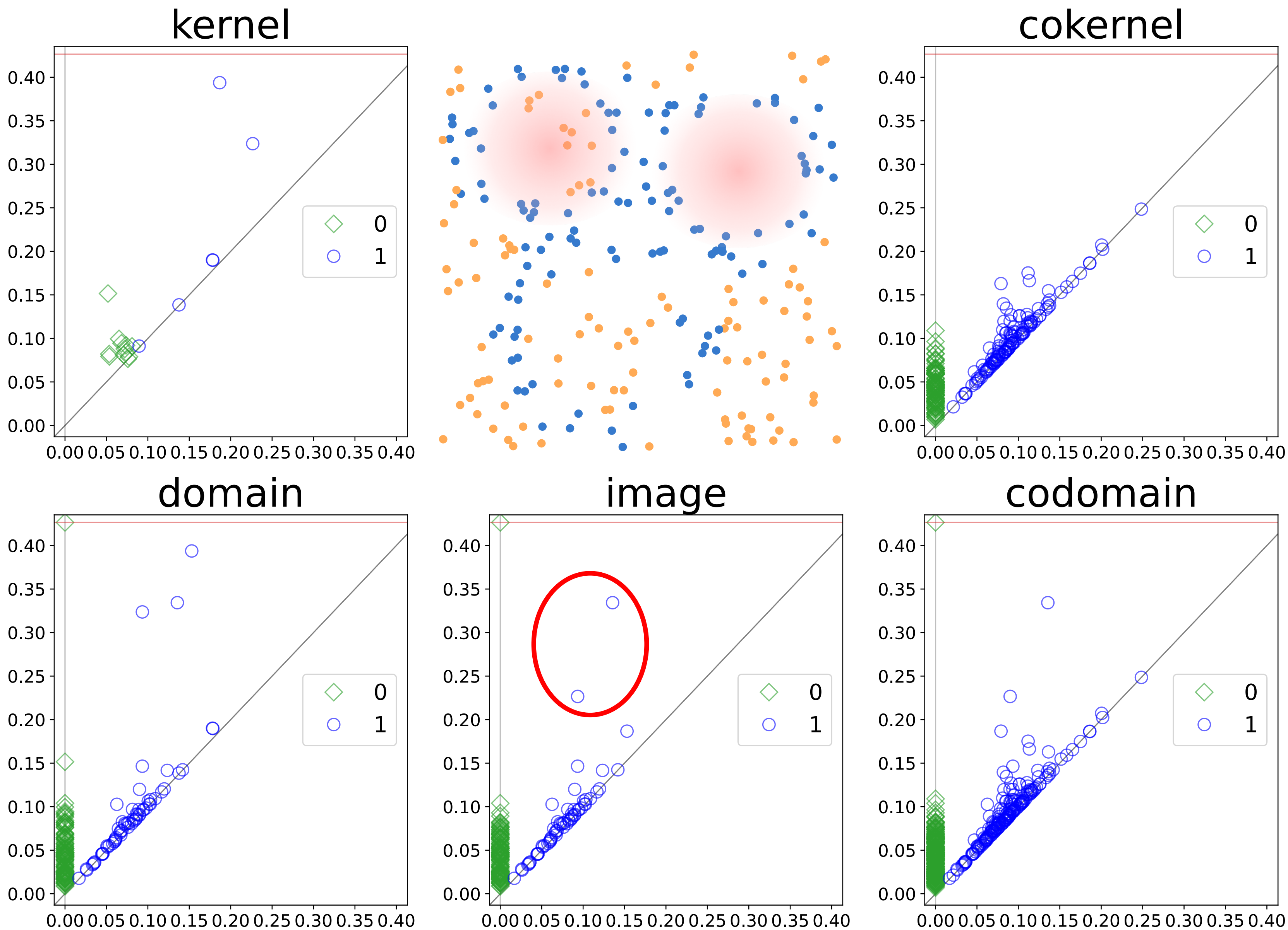

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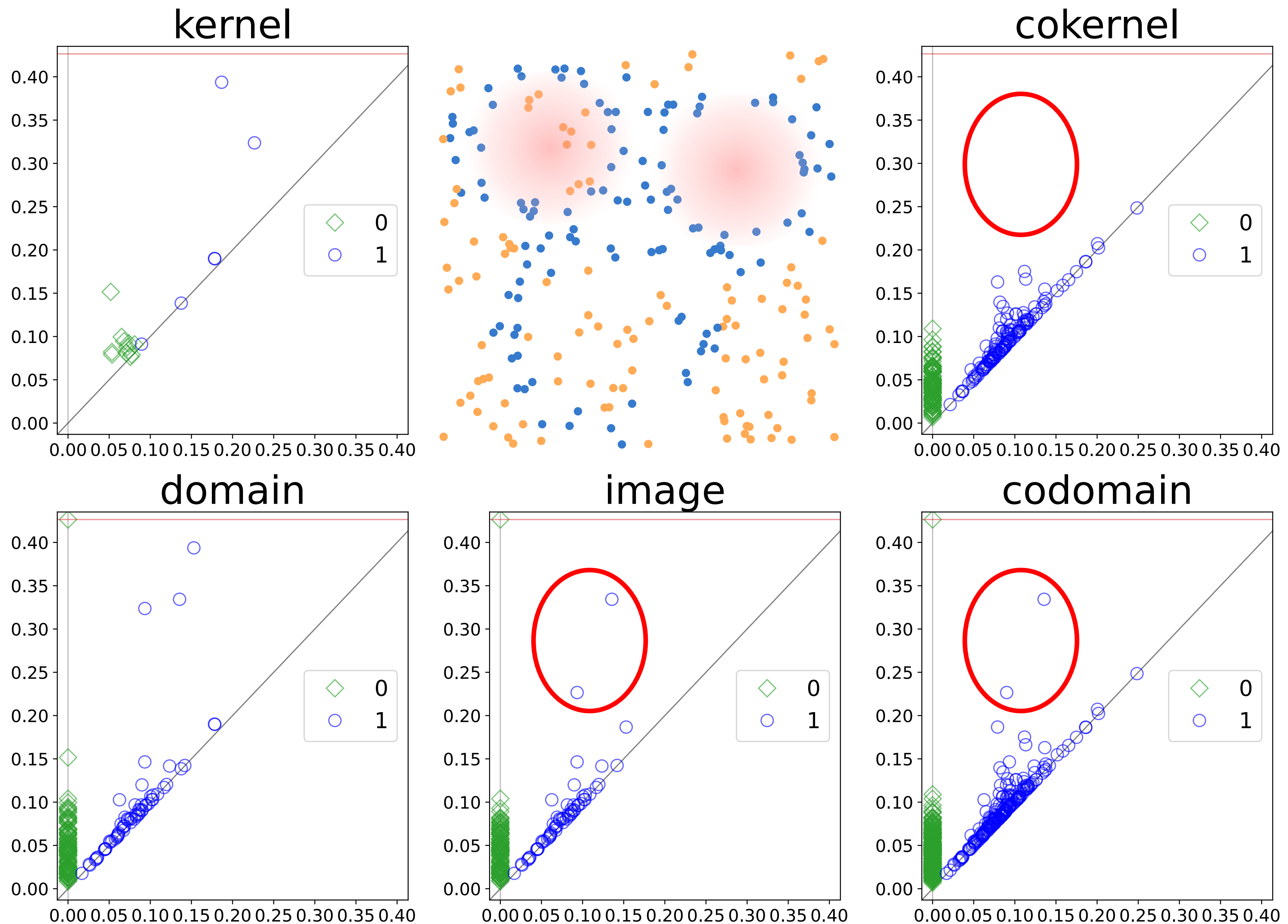

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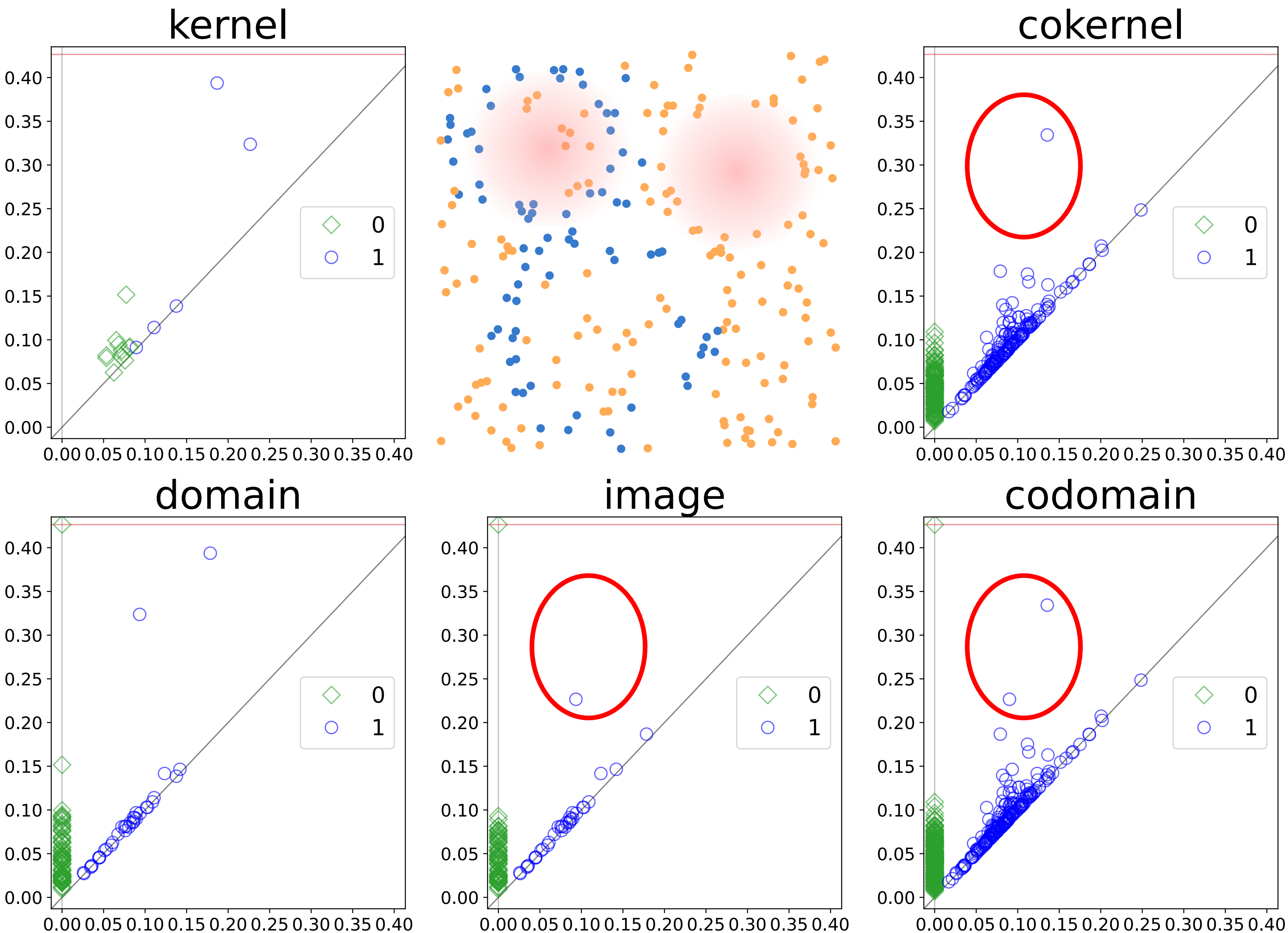
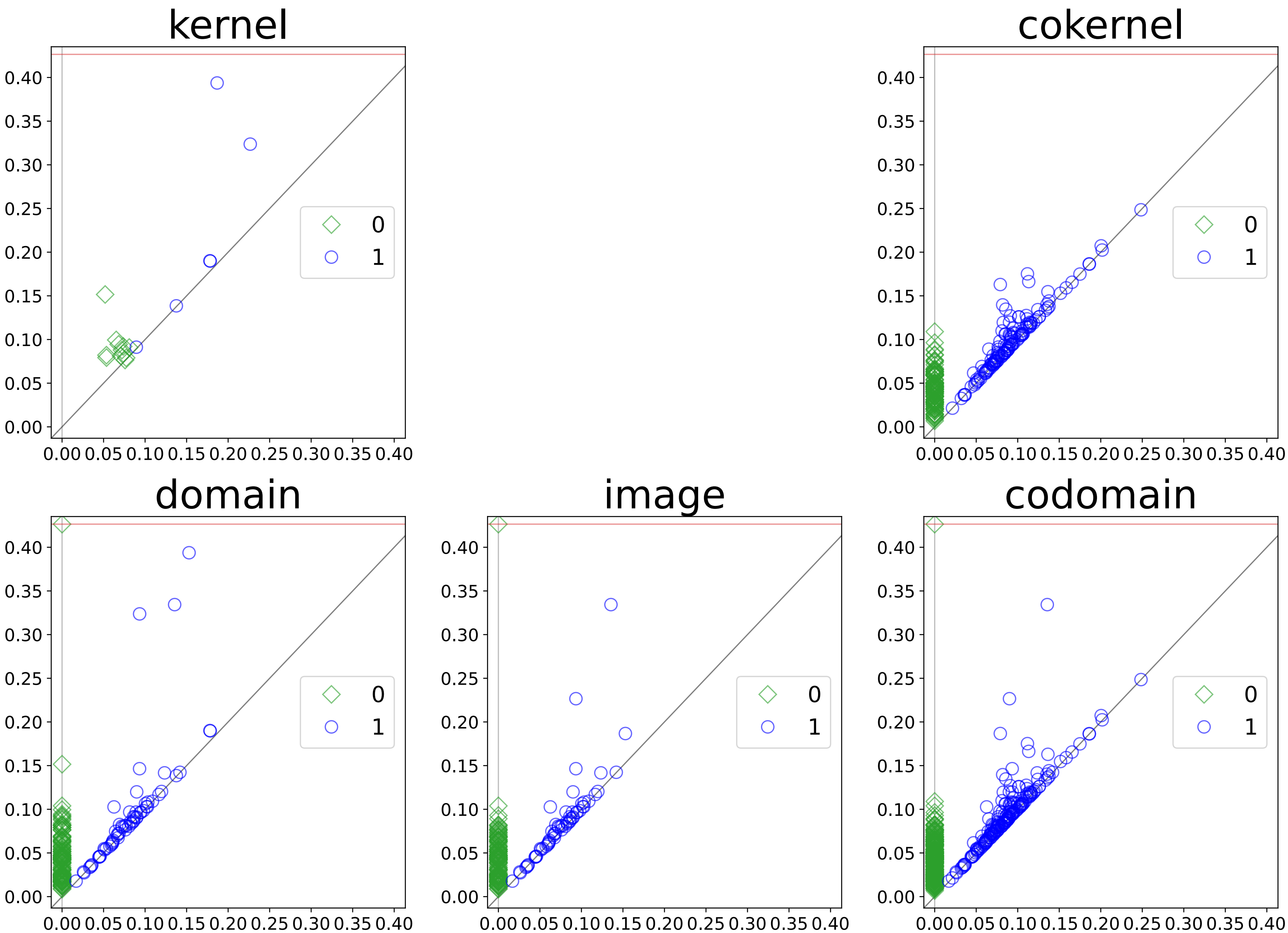


Short exact sequences in a six-pack



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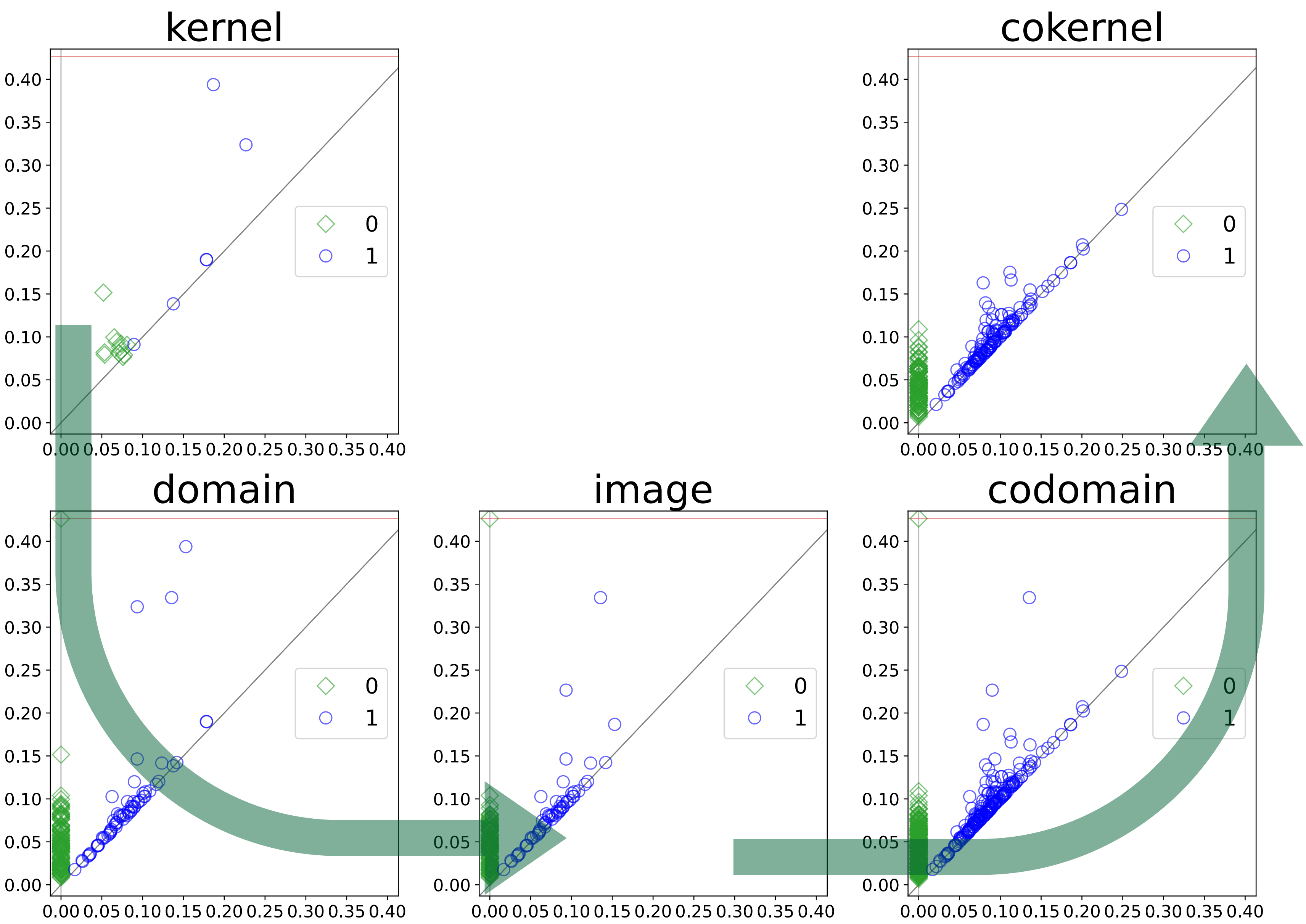

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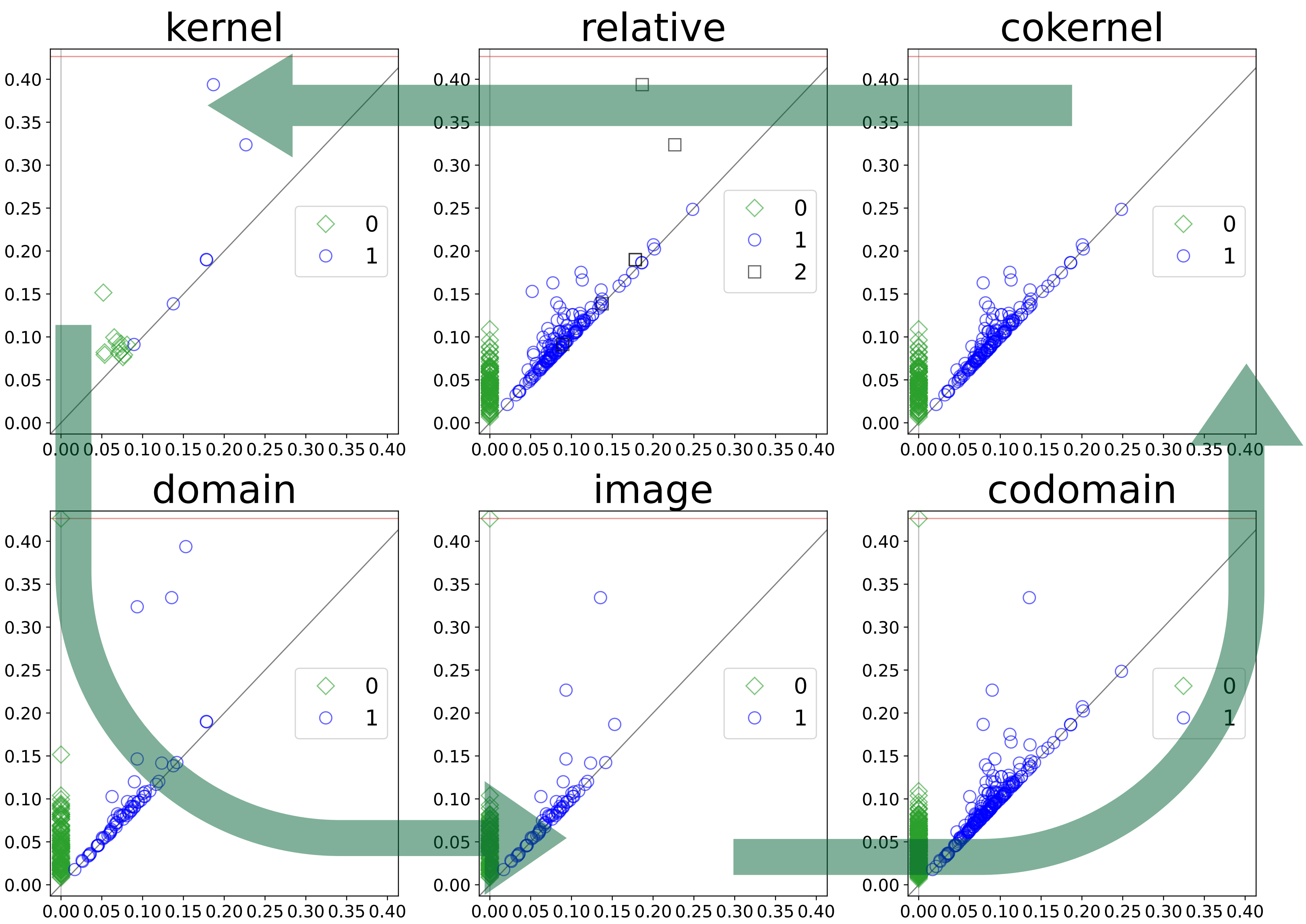

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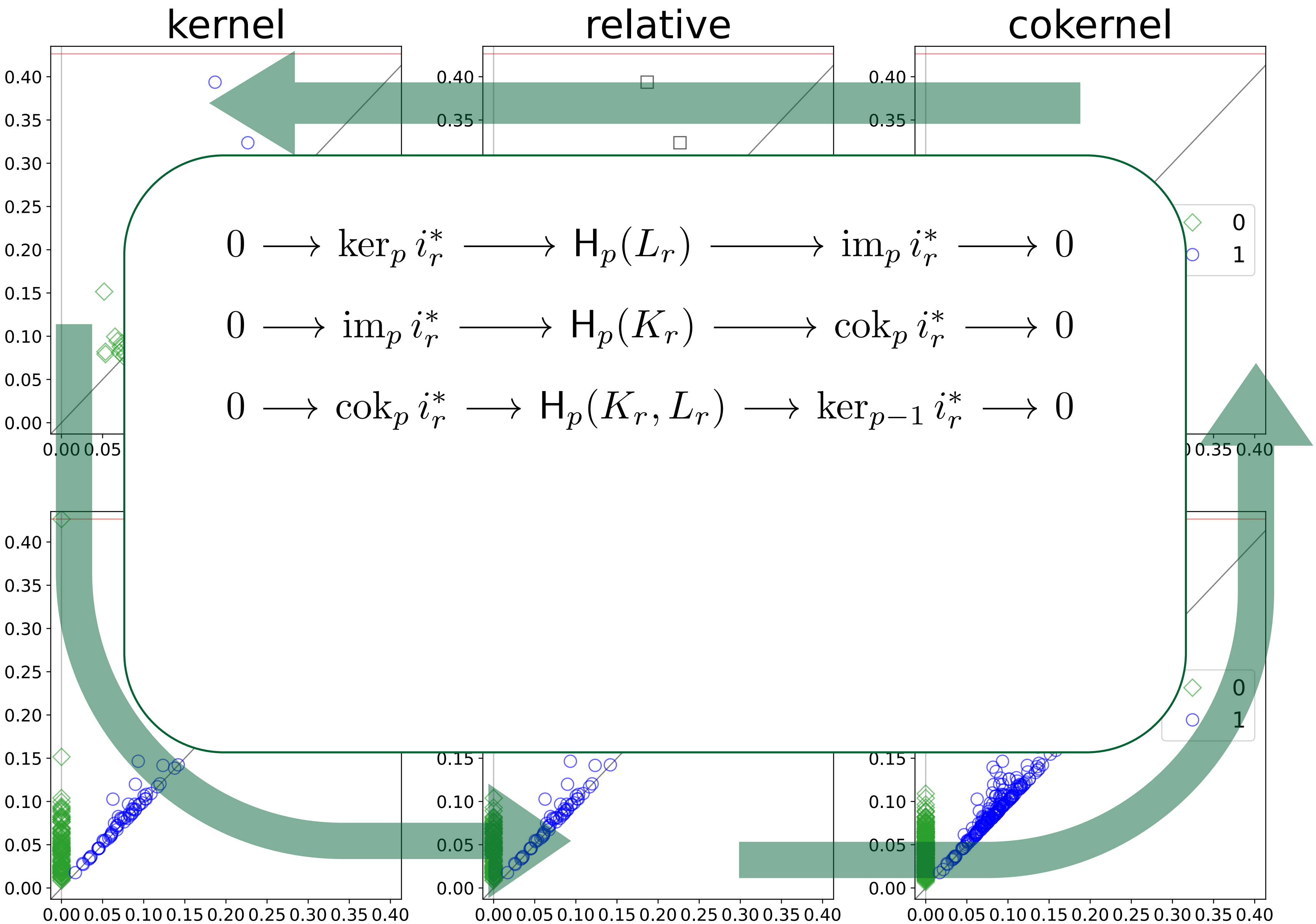

$$0 \longrightarrow \ker_p i_r^* \longrightarrow \mathsf{H}_p(L_r) \longrightarrow \operatorname{im}_p i_r^* \longrightarrow 0$$

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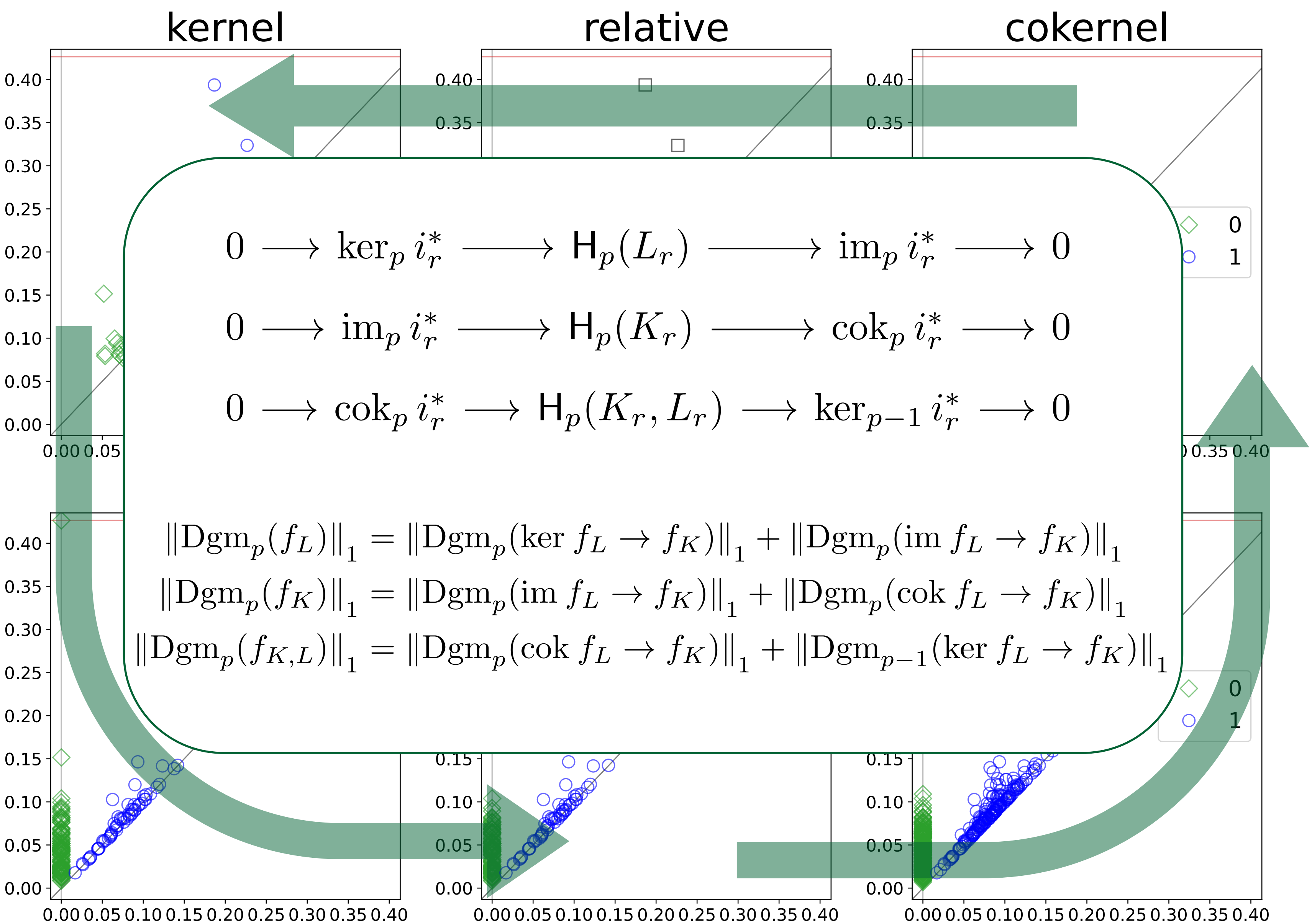

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$$\|\mathrm{Dgm}_p(f_L)\|_1 = \|\mathrm{Dgm}_p(\ker f_L \to f_K)\|_1 + \|\mathrm{Dgm}_p(\operatorname{im} f_L \to f_K)\|_1$$

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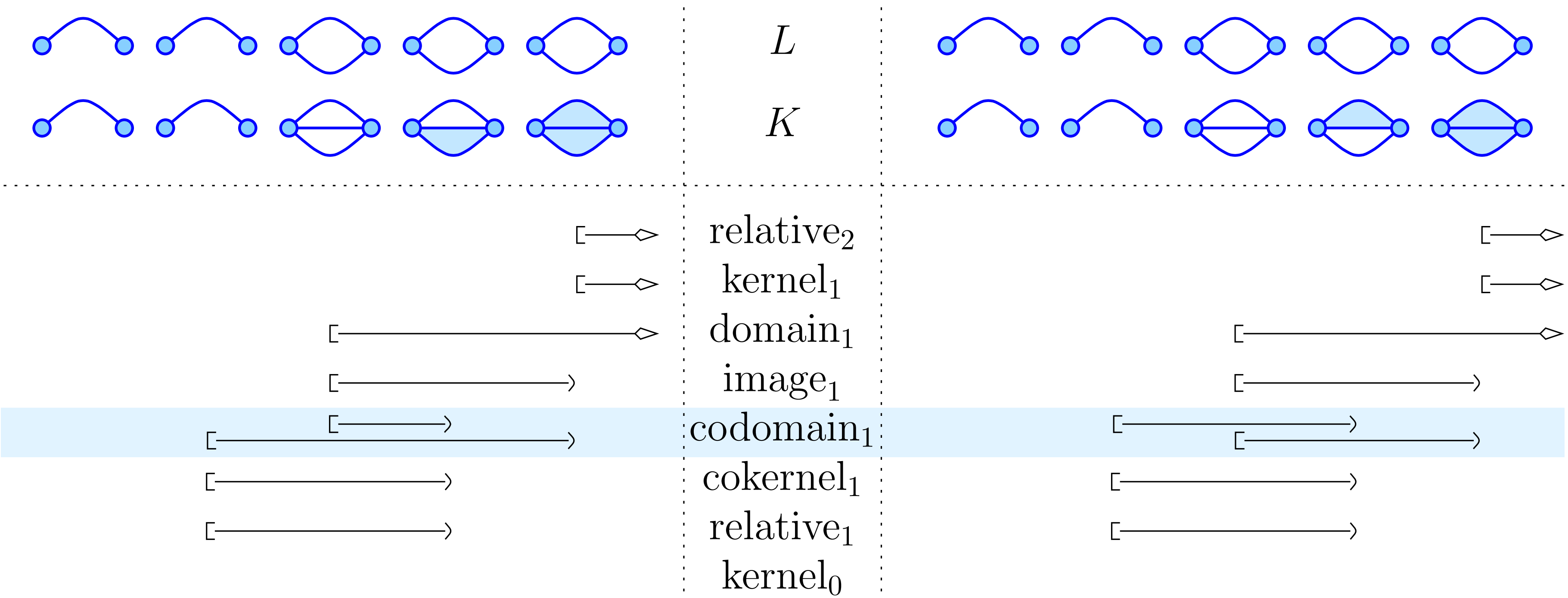
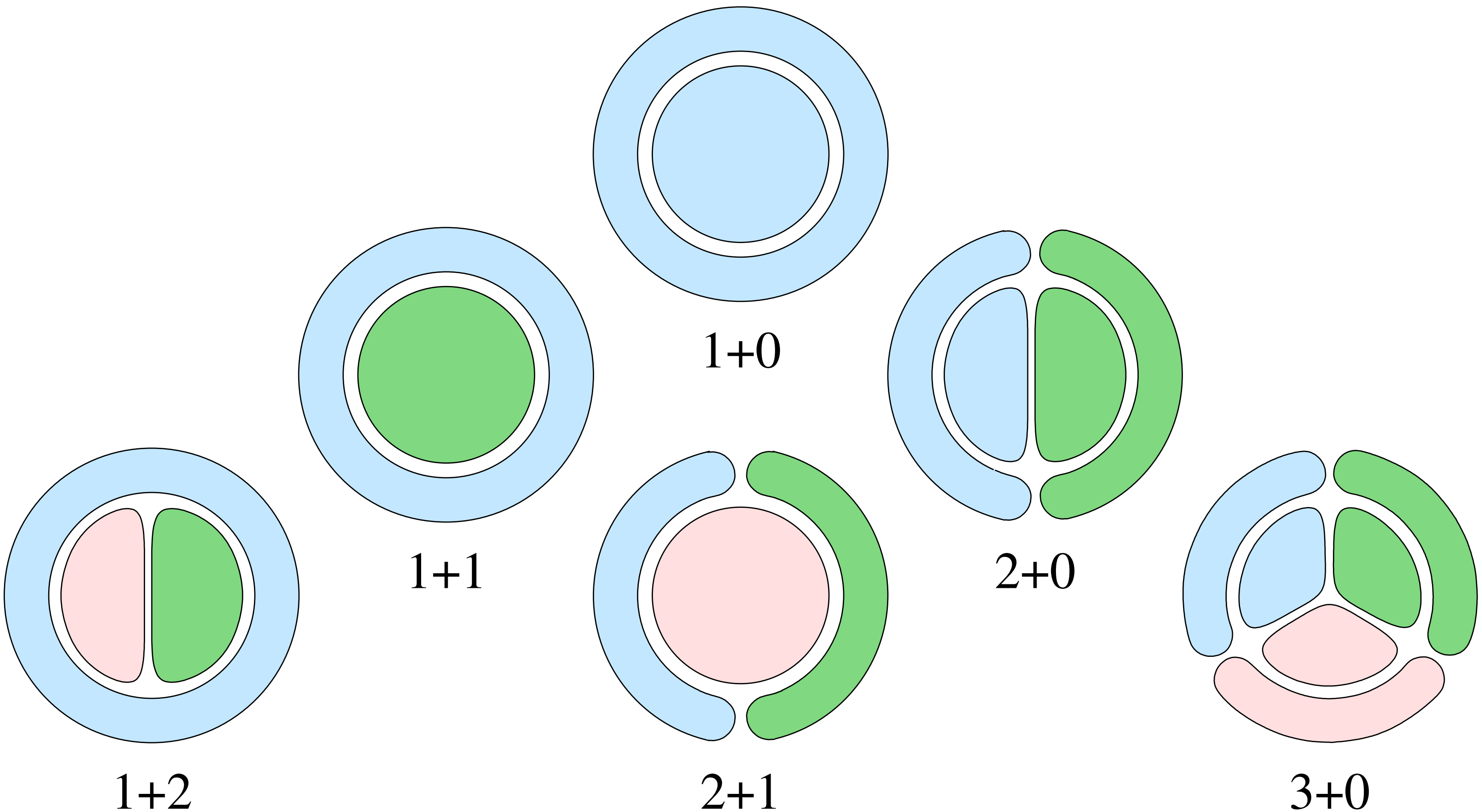
Five do not determine the sixth



More than two colors?



More than two colors?

- Six-pack is defined for some subcomplex
- A meaningful choice: k-chromatic subcomplex

$$L = \{\nu \in \mathrm{Del}(\chi) \mid \#\chi(\nu) \leq k\}$$

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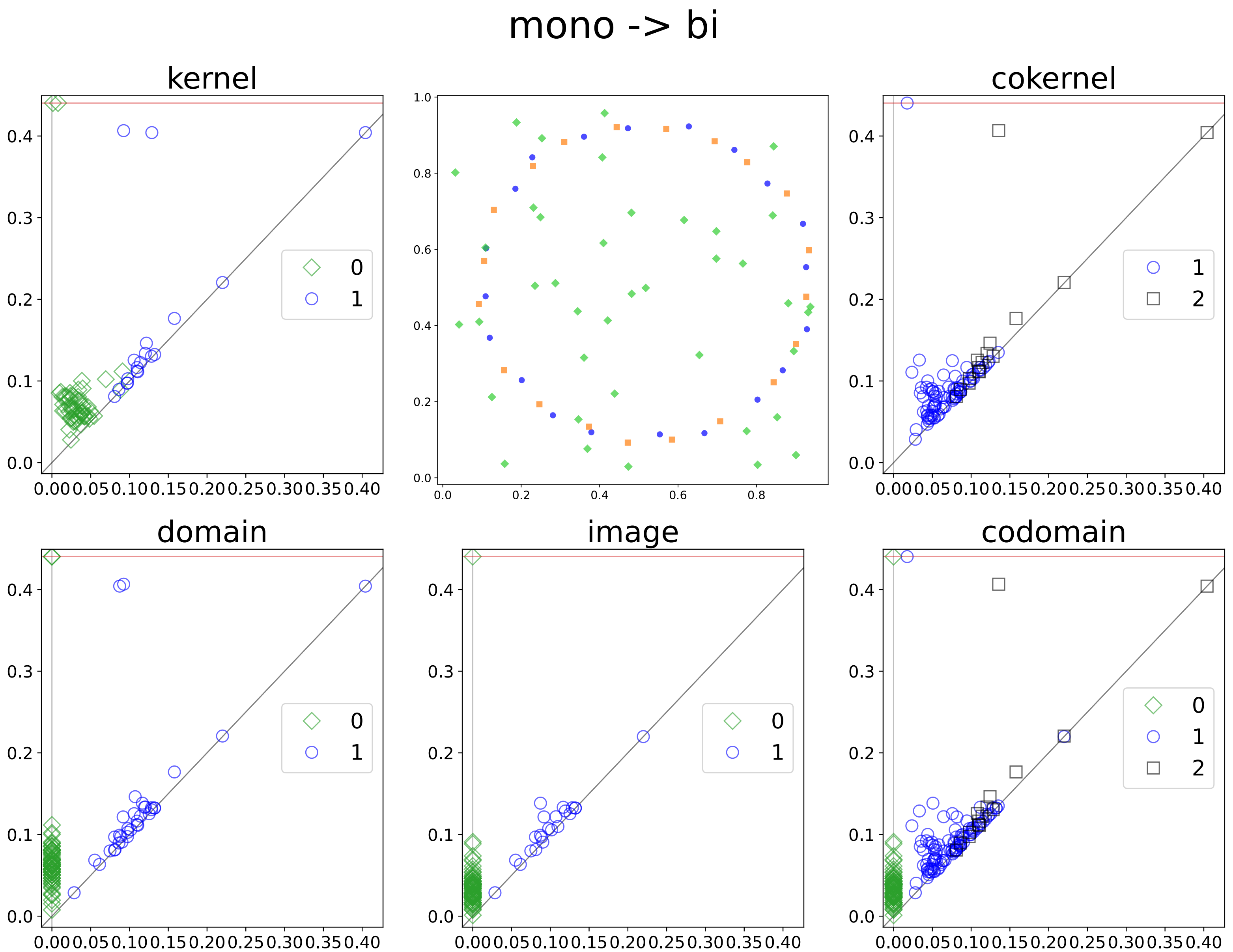
- Fourth option – relative:
  - bi-chr. / mono-chr. → everything / mono-chr.

3-color examples



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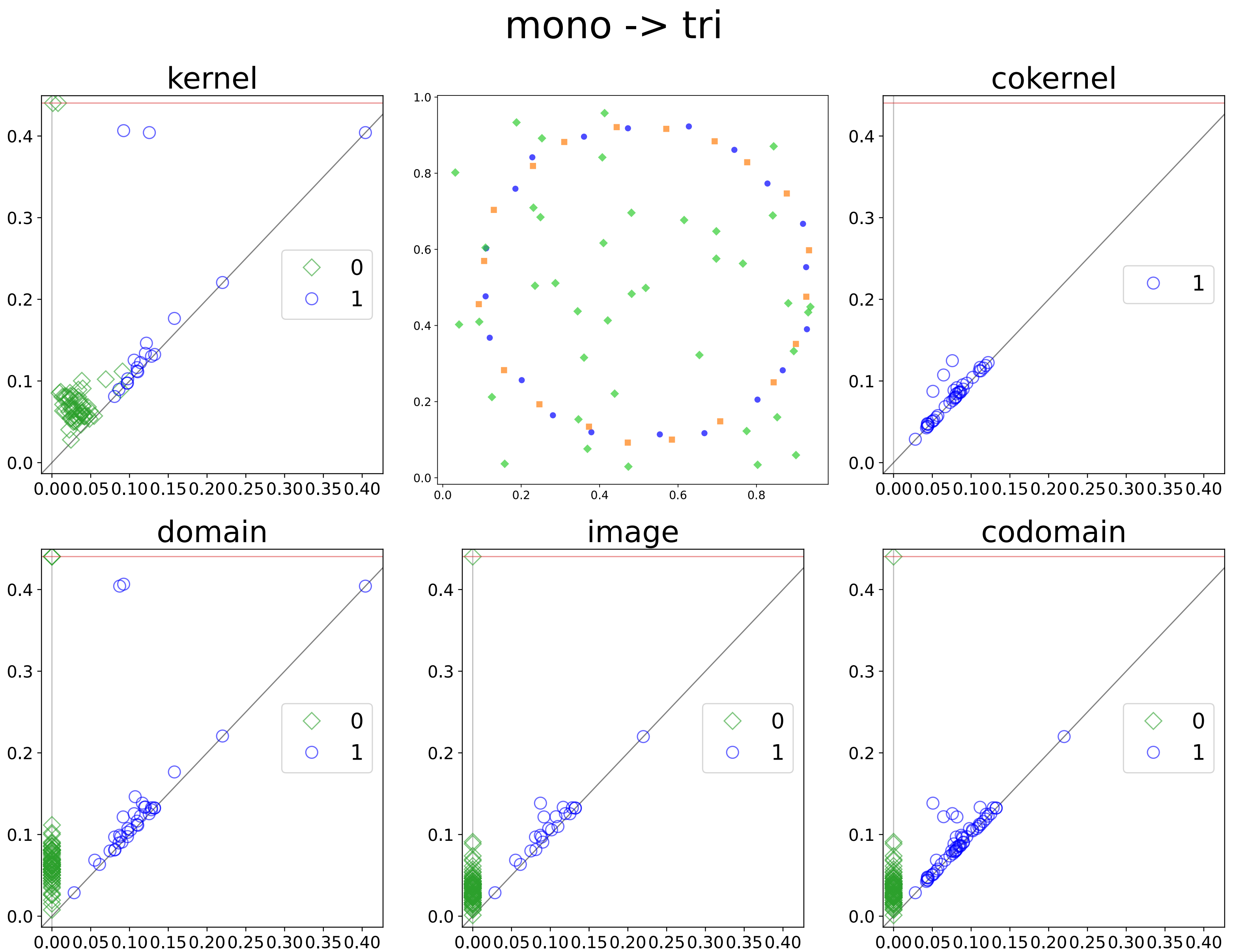

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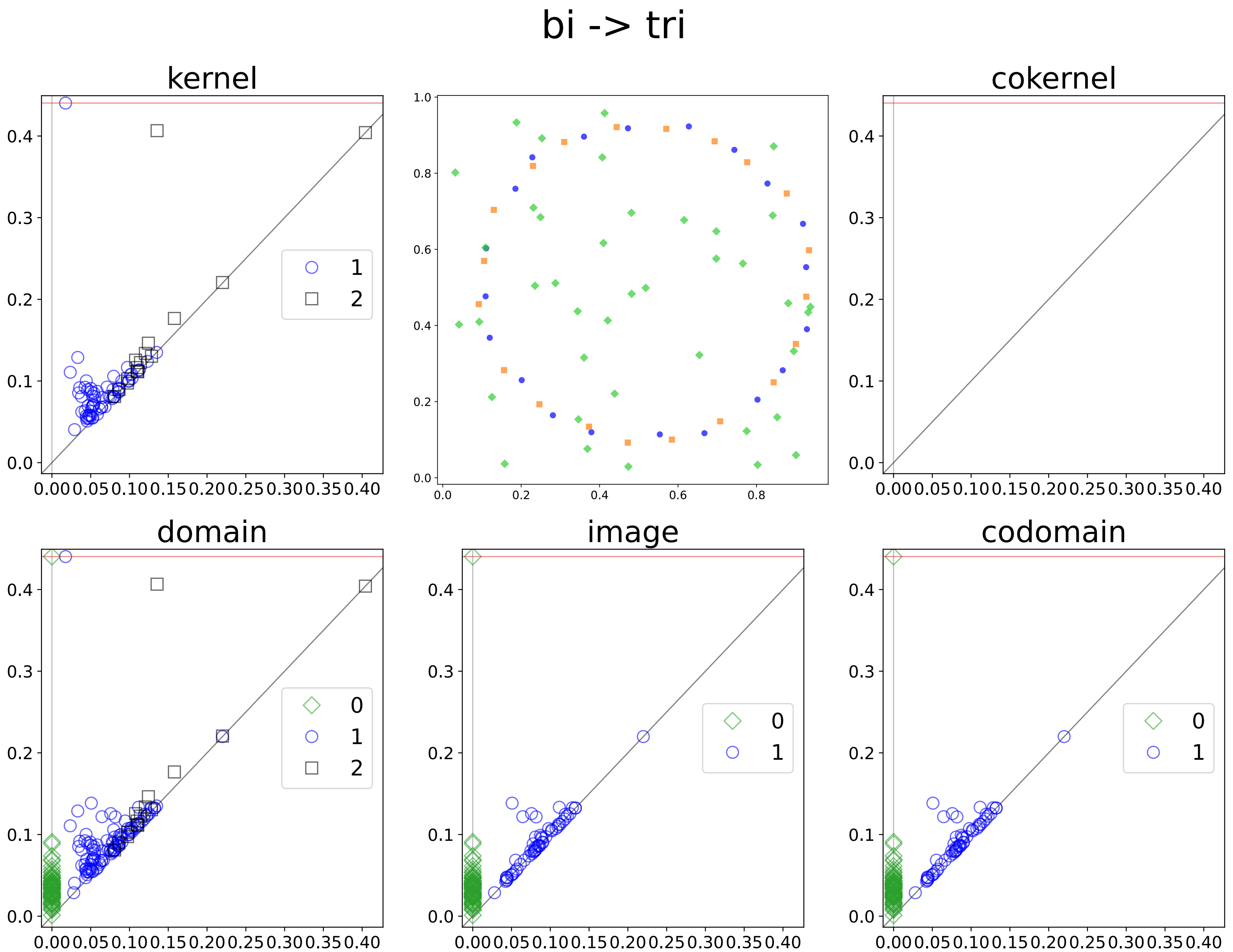

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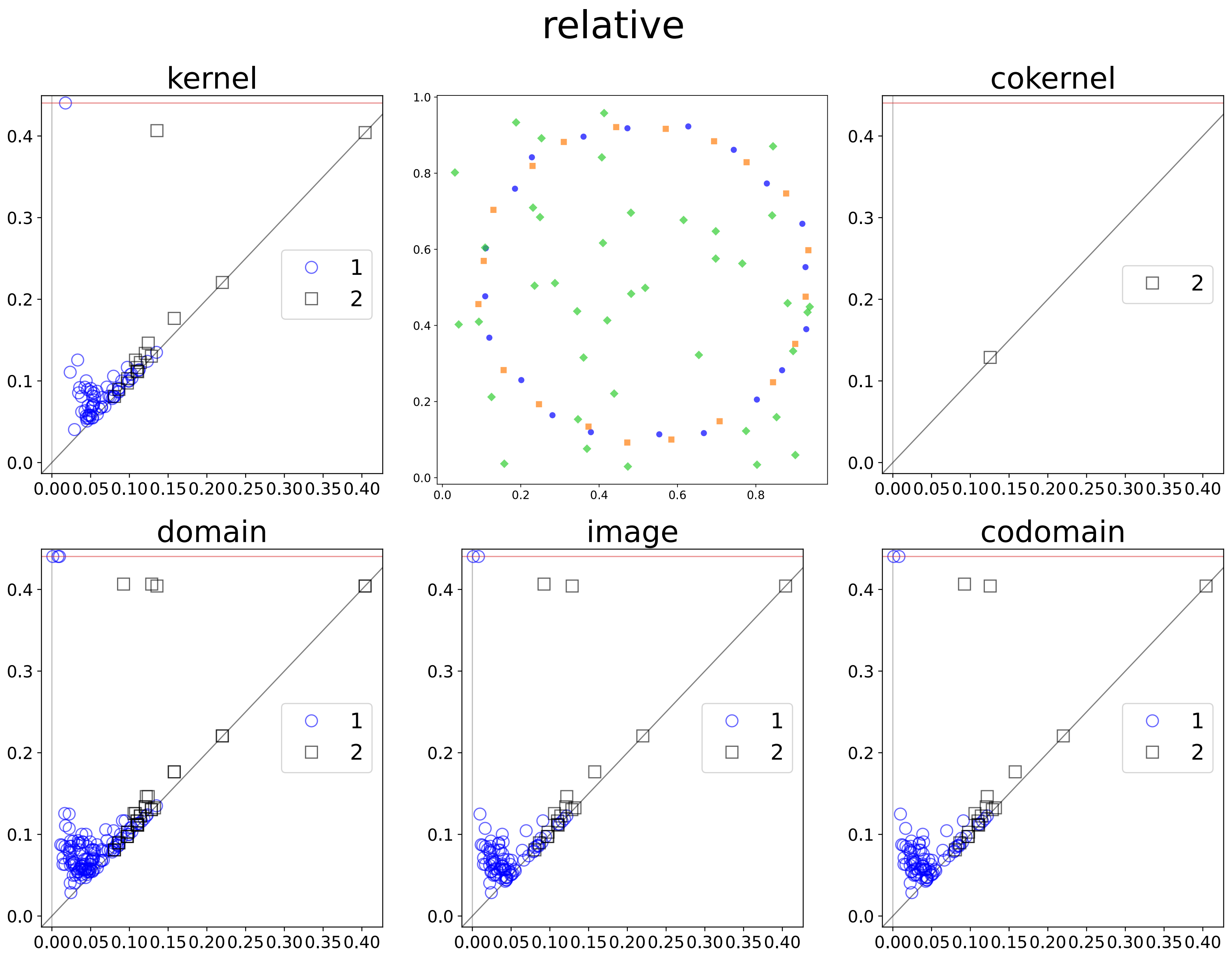

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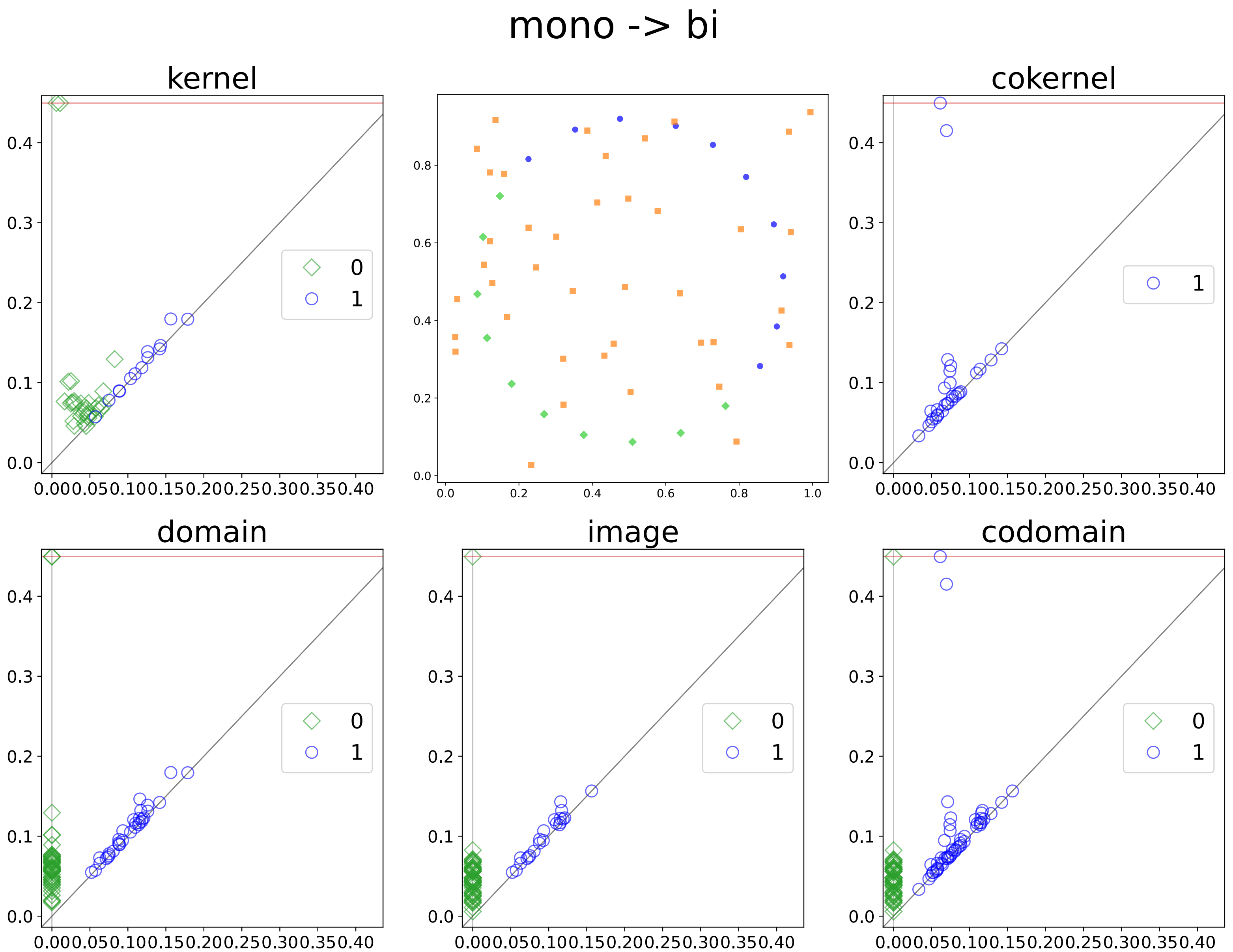

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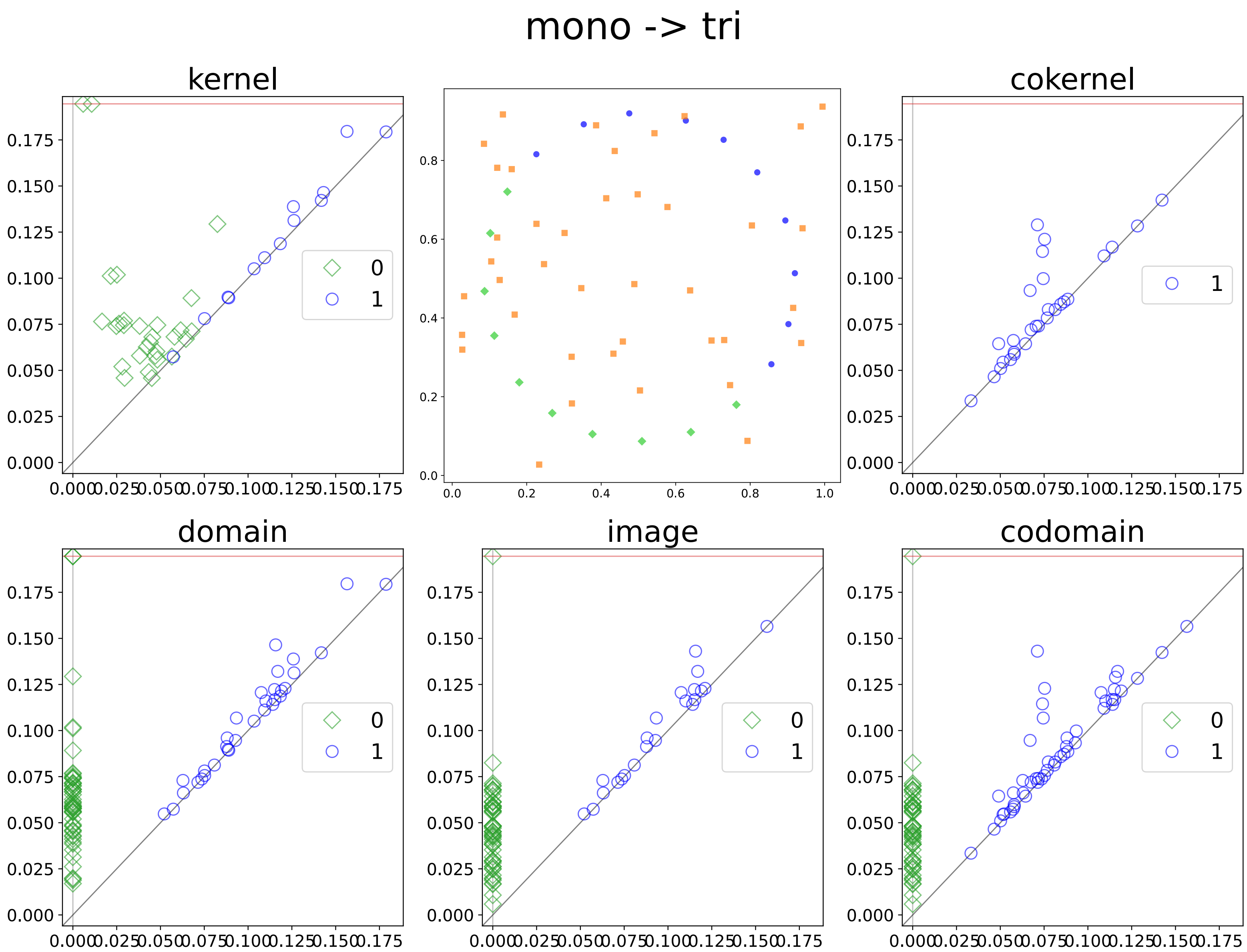

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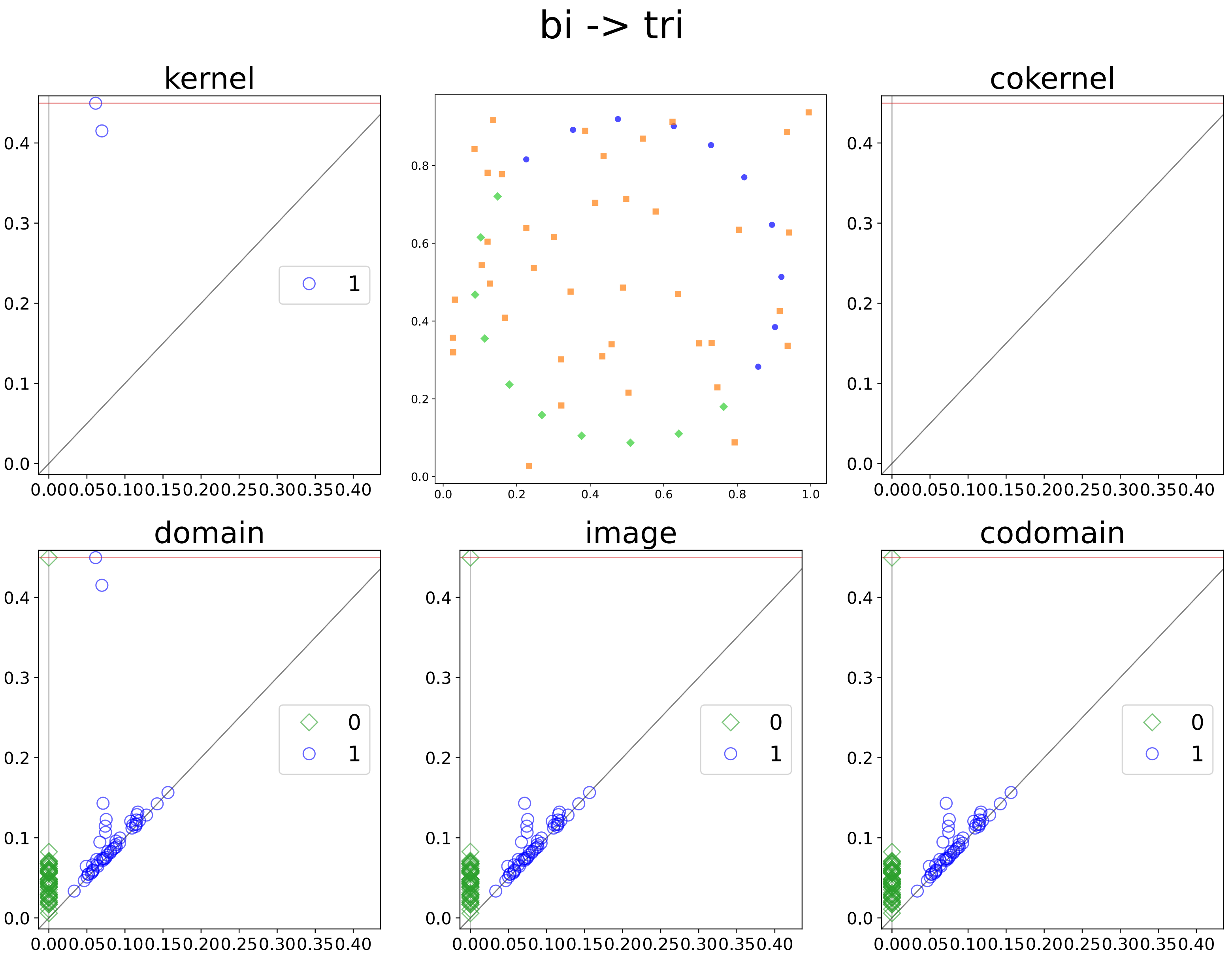

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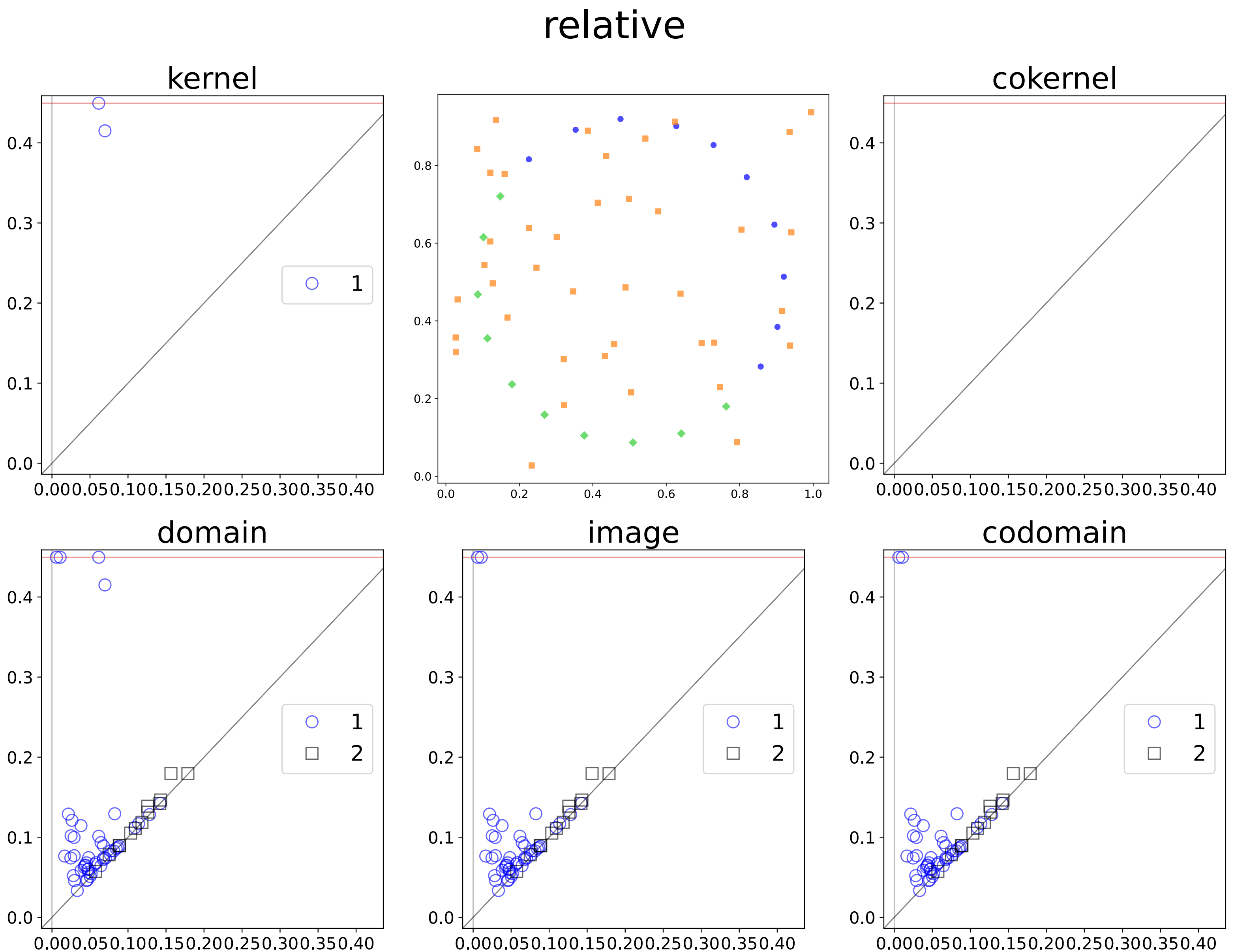

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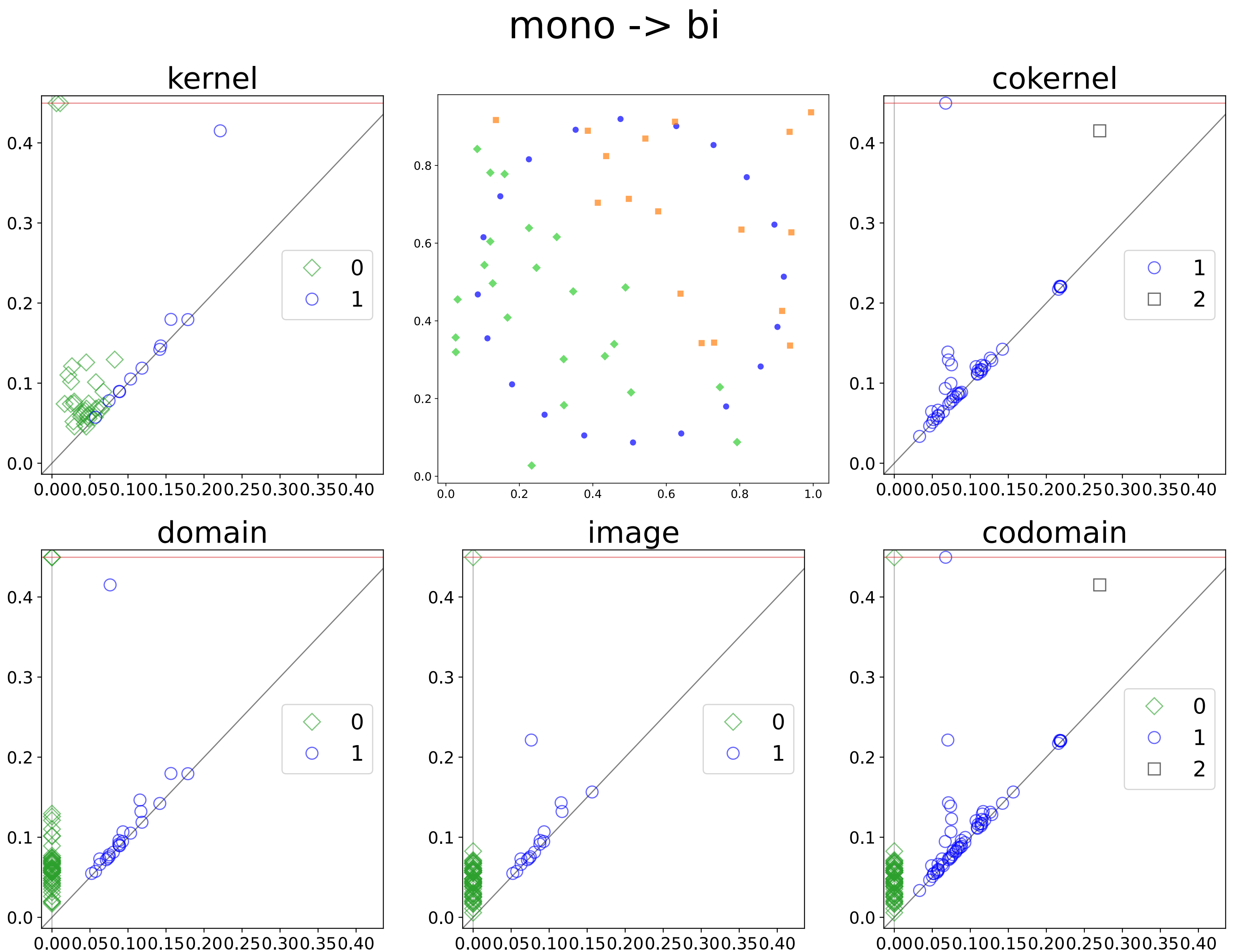

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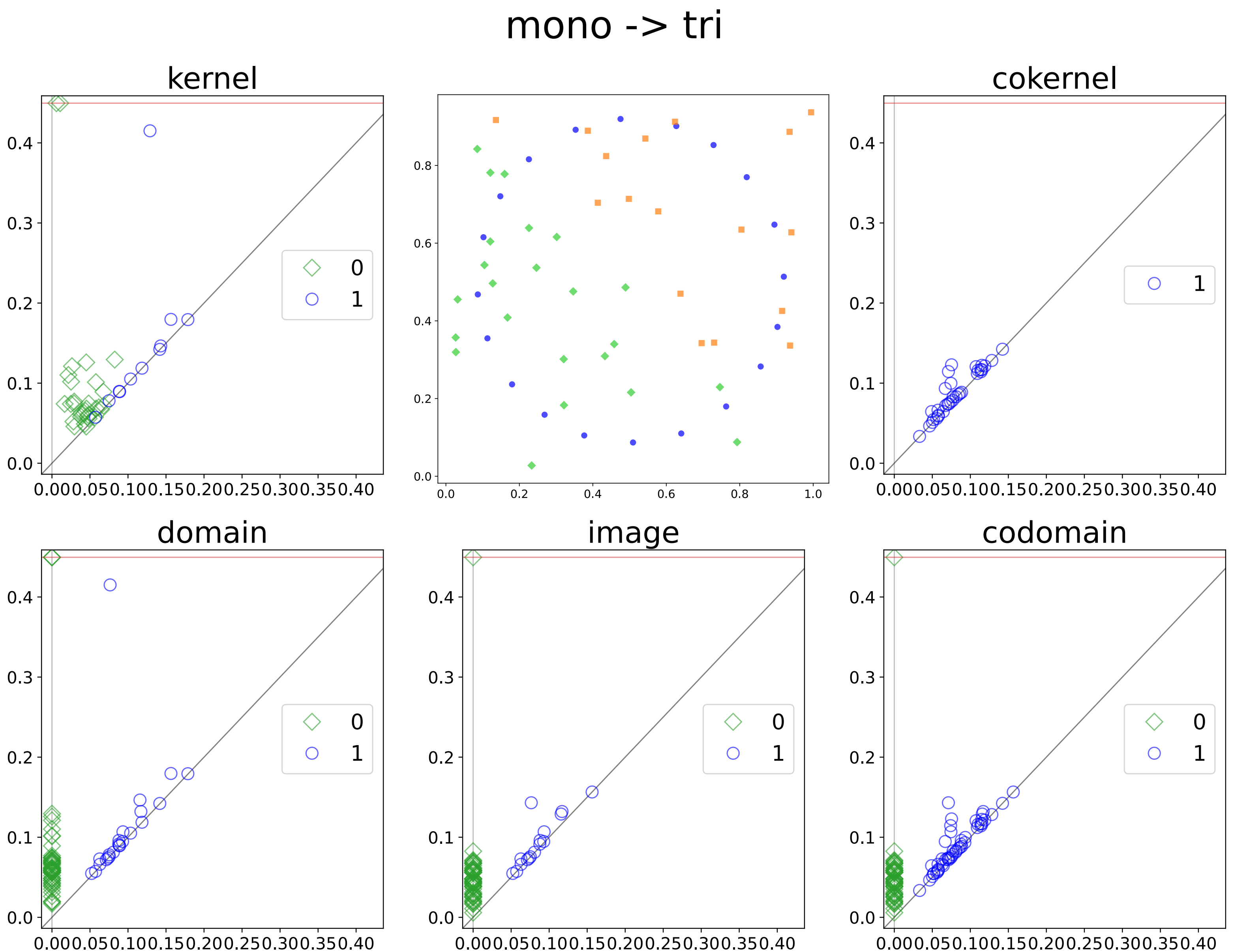

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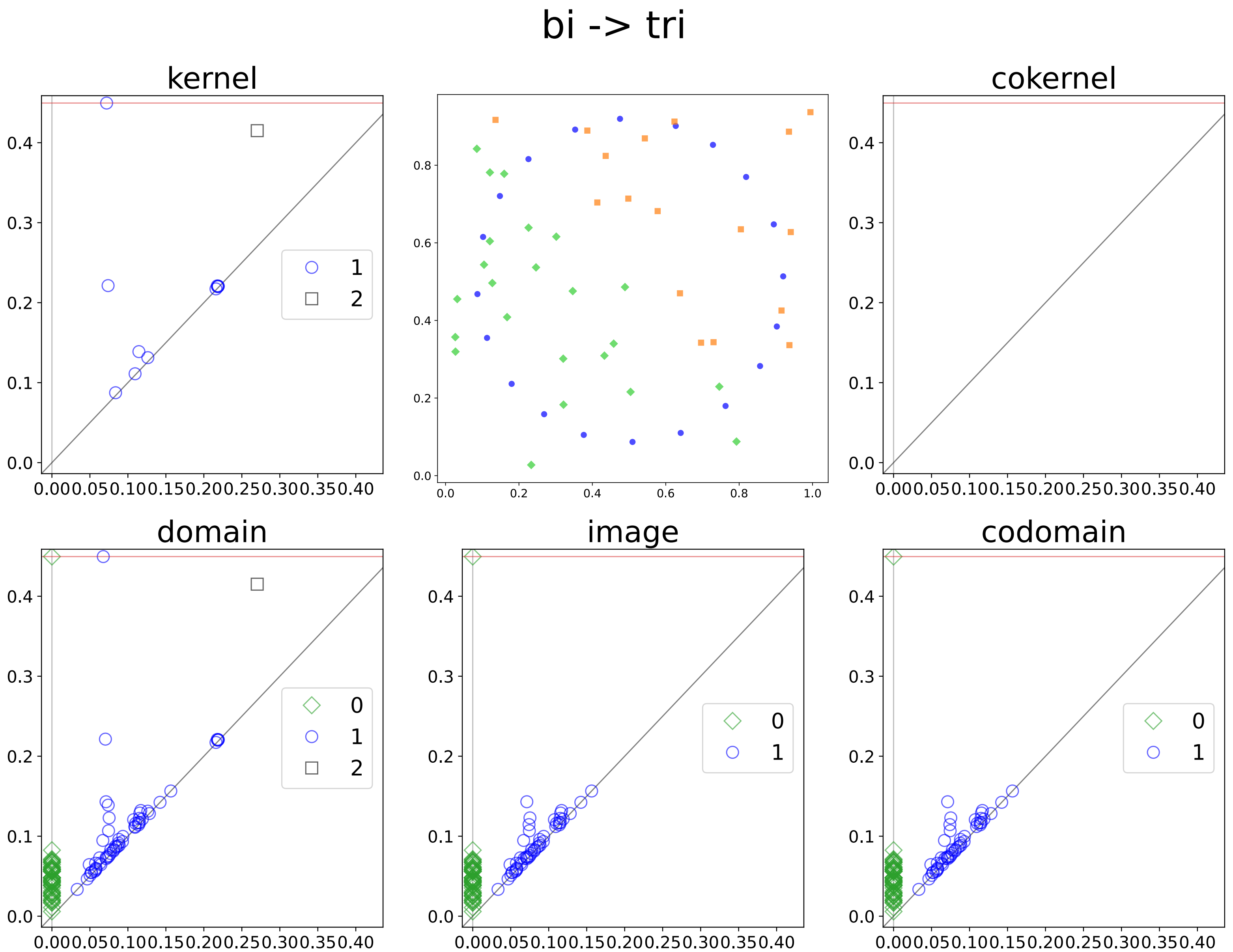

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